# OPTIMAL CONTINUOUS MANEUVERS FOR SATELLITE FORMATION RECONFIGURATION IN J2-PERTURBED ORBITS 

G. Di Mauro, ${ }^{\star}$ D. Spiller, ${ }^{\dagger}$ R. Bevilacqua ${ }^{\ddagger}$, and F. Curti§


#### Abstract

This paper focuses on the fuel-minimum in-plane spacecraft reconfiguration maneuver in $J_{2}$ perturbed near-circular orbits. The reconfiguration problem is posed as a nonlinear optimal control problem and it is solved by two techniques, namely the Mixed-integer Linear Programming and the Particle Swarm Optimization. The control is assumed to be a piecewise constant function and a linear dynamics model based on relative orbit element parameterization is used to derive the fueloptimal solution. Simulation results demonstrate the efficiency of both proposed methods, pointing out the performance in terms of computing time and accuracy.


## INTRODUCTION

In the recent years, there has been an increasing interest on the spacecraft formation flying concepts. In fact, the use of multiple spacecraft operating in a coordinated way allows improving the mission performance, while providing increased adaptability, versatility, and robustness ${ }^{12}$.

Among the various technical challenges involved in spacecraft formation flying, the capability to reconfigure the relative motion represents a key aspect that has been intensively studied over the last years ${ }^{1}$. The formation reconfiguration problem is defined as the achievement of a specific relative formation geometry in a defined time interval, given a general initial relative configuration. So far, many methods have been proposed to solve the aforementioned problem, ranging from the impulsive to the continuous control techniques. The impulsive approach for the satellite formation control has been widely discussed in many works. Vaddi et al. derived a two-impulse analytical solution for the formation establishment and reconfiguration problems, using the Gauss' variational equations (GVE) in terms of nonsingular elements ${ }^{2}$. Chernick et al. developed a closed-form scheme for the in-plane and out-of-plane reconfigurations problem in near-circular perturbed and eccentric unperturbed orbits, using the relative orbit elements (ROE) to parameterize the equations of relative motion ${ }^{15}$. The continuous methodology is implemented when the maneuverable satellites in the formation are equipped with low-thrust actuation system, generally preferred for its fuel consumption efficiency due to its high specific impulse. Lawn et al. used the input-shaping filtering theory to derive the continuous analytical control solution for the short-distance planar spacecraft rephasing and rendezvous maneuvering problems ${ }^{3}$. In further details, they exploited the Schweighart and Sedwick (SS) linear dynamics model to obtain the analytical control solution. More recently, the authors of this paper derived a fully analytical solution for in-plane reconfiguration with three tangential finite-time maneuvers by inverting the ROE-based linearized equations of relative motion. In addition, they proposed a semi-analytical approach to

[^0]solve the out-of-plane satellite formation control problem with a single finite-time maneuver ${ }^{11}$. Many numerical methods have been also investigated for the computation of the optimal reconfiguration maneuver using continuous low-thrust propulsion system. Richards et al. proposed fuel-optimal control algorithm by using the linear time-varying Clohessy-Wiltshire (CW) relative dynamics model. The trajectory optimization approach were based on the solution of a mixed-integer linear programming (MILP) problem ${ }^{4}$. Acikmese et al. presented a convex guidance algorithm for optimal formation reconfiguration with collision avoidance using CW equations. The collision avoidance constraints are imposed via separating planes between each pair of spacecraft. Moreover, a heuristic is introduced to choose these separating planes that leads to the convexification of the collision avoidance constraints ${ }^{5}$. Huntington et al. developed a nonlinear fuel-optimal configuration method for tetrahedral formation based on Gauss variational equations. The associated optimization problem is solved using Gauss pseudospectral method ${ }^{6}$. Massari et al. proposed a nonlinear low-thrust trajectory optimization method using a combination of parallel multiple shooting direct transcription and a barrier interior point method. They exploited a nonlinear dynamics model to describe the relative motion considering any kind of positional force field ${ }^{7}$.

This paper addresses the design of the fuel-minimum spacecraft formation reconfiguration strategy in near-circular $J_{2}$-perturbed orbits. In further details, the in-plane reconfiguration problem is investigated. Two different techniques, namely the Particle Swarm Optimization (PSO) (Reference 8) and the Mixed-Integer Linear Programming (MILP) (Reference 9), are proposed to solve the associated minimization problem. Both aforementioned approaches exploit a linear dynamics model based on relative orbit element (ROE) parameterization and its associated analytical solution to describe the relative motion.

The reminder of this paper is organized as follows. First, the ROE-based relative dynamics describing the satellite formation motion as well as its associated analytical solution is introduced, then the optimization problem related to the design of the fuel/optimal reconfiguration maneuvering strategy is presented. Second, the PSO and MILP are detailed. Finally, simulation results are presented for the validation of the proposed techniques.

## PROBLEM STATEMENT

This section aims at defining the optimal control problem associated to the design of the fuel-minimum strategy for the in-plane formation flying reconfiguration. First, the linear dynamics model describing the relative motion between two Earth orbiting satellites is presented, along with the corresponding analytical solution. The proposed dynamical model is formulated using the dimensionless relative orbit elements (ROE) defined by D'Amico in (Reference 10). It allows the inclusion of the $J_{2}$ effects as well as those due to the external accelerations. Finally, the acceleration control profile used in this work is presented.

## Relative Dynamics Model

The relative motion of a spacecraft (deputy) with respect to another one, referred to as chief, can be parameterized using the dimensionless relative orbit elements defined by ${ }^{10}$

$$
\delta \boldsymbol{\alpha}=\left[\begin{array}{c}
a_{d} / a_{c}-1  \tag{1}\\
\left(u_{d}-u_{c}\right)+\left(\Omega_{d}-\Omega_{c}\right) \mathrm{c}_{i_{c}} \\
e_{d} \mathrm{c}_{\omega_{d}}-e_{c} \mathrm{c}_{\omega_{c}} \\
e_{d} \mathrm{~s}_{\omega_{d}}-e_{c} \mathrm{~s}_{\omega_{c}} \\
i_{d}-i_{c} \\
\left(\Omega_{d}-\Omega_{c}\right) \mathrm{s}_{i_{c}}
\end{array}\right]=\left[\begin{array}{c}
\delta a \\
\delta \lambda \\
\delta e_{x} \\
\delta e_{y} \\
\delta i_{x} \\
\delta i_{y}
\end{array}\right]
$$

where $a, e, i, \omega, \Omega$, and $M$ represent the classical Keplerian elements, with the subscripts $c$ and $d$ standing for chief and deputy, respectively. In this parameterization, $a$ is the relative semi-major axis, $\delta \lambda$ is the relative mean longitude, $\delta \boldsymbol{e}$ indicates the relative eccentricity vector, and $\delta \boldsymbol{i}$ is the relative inclination vector. Note that the symbols $\mathrm{s}_{(.)}$and $\mathrm{c}_{(.)}$indicate the trigonometric functions $\sin ($.$) and \cos ($.$) , respectively.$

As discussed by the authors in (Reference 11), the averaging theory (Reference 12) can be used to derive the variation of mean ROE due to the Earth's oblateness $J_{2}$. Moreover, assuming that the mean orbit elements are reasonably approximated by the corresponding osculating ones as the Jacobian of the osculating-mean
mapping is approximately a $6 \times 6$ identity matrix with the off-diagonal terms being of order $J_{2}$ or smaller (Reference 13), the well-known Gauss Variational Equations (GVE) (Reference 14) can be exploited to determine the change of the mean $\operatorname{ROE}$ due to the continuous control acceleration $\boldsymbol{F}(t) \in \mathbb{R}^{3}$. Hence, the set of nonlinear differential equations describing the mean relative motion under the effects of $J_{2}$ perturbing acceleration and the continuous control acceleration acting on the deputy is ${ }^{11}$

$$
\delta \dot{\boldsymbol{\alpha}}=\left[\begin{array}{c}
\mathbf{0}  \tag{2}\\
n_{d}\left(\boldsymbol{\alpha}_{c}, \delta \boldsymbol{\alpha}\right)-n_{c}\left(\boldsymbol{\alpha}_{c}\right) \\
\mathbf{0}_{4 \times 1}
\end{array}\right]+\boldsymbol{\sigma}_{J_{2}}\left(\boldsymbol{\alpha}_{c}, \delta \boldsymbol{\alpha}\right)+\boldsymbol{\sigma}_{F}\left(\boldsymbol{\alpha}_{c}, \delta \boldsymbol{\alpha}, \boldsymbol{F}(t)\right)=\xi\left(\boldsymbol{\alpha}_{c}, \delta \boldsymbol{\alpha}, \boldsymbol{F}(t)\right)
$$

where

$$
\boldsymbol{\sigma}_{J_{2}}\left(\boldsymbol{\alpha}_{c}, \delta \boldsymbol{\alpha}\right)=\left[\begin{array}{c}
0 \\
\left(\eta_{d} P_{d} K_{d}-\eta_{c} P_{c} K_{c}\right)+\left(K_{d} Q_{d}-K_{c} Q_{c}\right)-2\left(K_{d} \mathrm{c}_{i_{d}}-K_{c} \mathrm{c}_{i_{c}}\right) \mathrm{c}_{i_{c}}  \tag{4}\\
-e_{d} \mathbf{s}_{\omega_{d}} K_{d} Q_{d}+e_{c} \mathbf{s}_{\omega_{c}} K_{c} Q_{c} \\
e_{d} \mathbf{c}_{\omega_{d}} K_{d} Q_{d}-e_{c} \mathbf{s}_{\omega_{c}} K_{c} Q_{c} \\
0 \\
-2\left(K_{d} \mathrm{c}_{i_{d}}-K_{c} \mathbf{c}_{i_{c}}\right) \mathrm{s}_{i_{c}}
\end{array}\right]
$$

In Eq. (3) the quantities $K_{(.)}, Q_{(.)}, P_{(.)}$, and $\eta_{(.)}$(the subscript " $($.$) " stands for c$ or $d$ ) are defined as follows

$$
\begin{gather*}
K_{(.)}=\frac{\gamma n_{(.)}}{a_{(.)}^{2} \eta_{(.)}^{4}} \quad \eta_{(.)}=\sqrt{1-e_{j}^{2}} \quad n_{(.)}=\sqrt{\frac{\mu_{\oplus}}{a_{(.)}^{3}}}  \tag{5}\\
Q_{(.)}=5 \cos \left(i_{(.)}\right)^{2}-1 \quad P_{(.)}=3 \cos \left(i_{(.)}\right)^{2}-1
\end{gather*} \quad \gamma=\frac{3}{4} J_{2} R_{E}^{2} .
$$

where $J_{2}$ indicates the second spherical harmonic of the Earth's geopotential $\left(J_{2}=1.082 \times 10^{-3}\right), R_{E}$ the Earth's equatorial radius $\left(R_{E}=6378.13 \mathrm{~km}\right)$ and $\mu_{\oplus}$ the Earth gravitational parameter ( $\mu_{\oplus}=$ $\left.398600.4415 \mathrm{~km}^{3} / \mathrm{s}^{2}\right)$. The individual terms of the control influence matrix $\boldsymbol{\Gamma}_{F}\left(\boldsymbol{\alpha}_{d}\right)$ in Eq. (4) are listed in Appendix A. Performing a first-order Taylor expansion of the nonlinear function $\boldsymbol{\xi}\left(\boldsymbol{\alpha}_{c}, \delta \boldsymbol{\alpha}, \boldsymbol{F}(t)\right.$ ) in Eq. (2) around the chief orbit (i.e., $\delta \boldsymbol{\alpha}=0$ and $\boldsymbol{F}=0$ ) and assuming that the chief is moving on a near-circular orbit (i.e. $e_{c} \rightarrow 0$ ) yield the following linear dynamics model

$$
\begin{equation*}
\left.\left.\delta \dot{\boldsymbol{\alpha}}(t)=\frac{\partial \xi}{\partial \delta \boldsymbol{\xi}}\right]_{\substack{\delta \boldsymbol{\alpha}=\mathbf{0} \\ \boldsymbol{F}}} \delta \boldsymbol{\alpha}(t)+\frac{\partial \xi}{\partial \boldsymbol{F}}\right]_{\substack{\delta \boldsymbol{\alpha}=\mathbf{0}}} \boldsymbol{F}=\boldsymbol{A}_{N C} \delta \boldsymbol{\alpha}(t)+\boldsymbol{B}_{N C}(t) \boldsymbol{F}(t) \tag{6}
\end{equation*}
$$

where the plant and input matrices $\boldsymbol{A}_{N C}$ and $\boldsymbol{B}_{N C}$ respectively are

$$
\boldsymbol{A}_{N C}=\left[\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0  \tag{7}\\
-\Lambda_{c} & 0 & 0 & 0 & -K_{c} F_{c} S_{c} & 0 \\
0 & 0 & 0 & -K_{c} Q_{c} & 0 & 0 \\
0 & 0 & K_{c} Q_{c} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{7 K_{c} S_{c}}{2} & 0 & 0 & 0 & 2 K_{c} T_{c} & 0
\end{array}\right]
$$

$$
\boldsymbol{B}_{N C}\left(u_{c}\right)=\frac{1}{n_{c} a_{c}}\left[\begin{array}{ccc}
0 & 2 & 0  \tag{8}\\
-2 & 0 & 0 \\
\mathrm{~s}_{u_{c}} & 2 \mathrm{c}_{u_{c}} & 0 \\
-\mathrm{c}_{u_{c}} & 2 \mathrm{~s}_{u_{c}} & 0 \\
0 & 0 & \mathrm{~s}_{u_{c}} \\
0 & 0 & \mathrm{c}_{u_{c}}
\end{array}\right]
$$

In Eq. (7) the following substitutions are applied for clarity

$$
\begin{equation*}
F_{c}=4+3 \eta_{c}, E_{c}=1+\eta_{c}, S_{c}=\sin \left(2 i_{c}\right), T_{c}=\sin \left(i_{c}\right)^{2}, \Lambda_{c}=\frac{3}{2} n_{c}+\frac{7}{2} E_{c} K_{c} P_{c} . \tag{9}
\end{equation*}
$$

The term $u_{c}=\omega_{c}+M_{c}$ in Eq. (8) indicates the mean argument of latitude of the chief orbit at the instant $t$ and is related to the time $t$ through the following expression

$$
\begin{equation*}
u_{c}=u_{0}+W_{c}\left(t-t_{0}\right) \tag{10}
\end{equation*}
$$

where $W_{c}=n_{c}+K_{c} Q_{c}+\eta_{c} K_{c} P_{c}$ (Reference 15) whereas $u_{0}=u_{c}\left(t_{0}\right)$. It is worth noting that the variables $t$ and $u_{c}$ are considered interchangeable as they are linearly related through the Eq. (10). As discussed in (Reference 11), given the linear differential equations (6)-(8) describing the relative dynamics, the mean ROE at the time $t$ can be determined through the following relationship ${ }^{11}$

$$
\begin{equation*}
\delta \boldsymbol{\alpha}(t)=\boldsymbol{\Phi}_{N C}\left(u_{c}, u_{0}\right) \delta \boldsymbol{\alpha}_{0}+\boldsymbol{\Psi}_{N C}\left(u_{c}, u_{0}\right) \boldsymbol{F}\left(u_{c}\right) \tag{11}
\end{equation*}
$$

where $\delta \boldsymbol{\alpha}_{0}=\delta \boldsymbol{\alpha}\left(t_{0}\right)$ is the mean ROE at the initial instant $t_{0}, \boldsymbol{F}\left(u_{c}\right) \in \mathbb{R}^{3}$ is a continuous piecewise constant function (see next section for more details). $\boldsymbol{\Phi}_{N C}\left(u_{c}, u_{0}\right)$ and $\boldsymbol{\Psi}_{N C}\left(u_{c}, u_{0}\right)$ denote the state transition matrix and the convolution matrix, respectively, associated with the linear dynamics system (6) and have the following form

$$
\begin{align*}
& \boldsymbol{\Phi}_{N C}\left(u_{c}, u_{0}\right)=\left[\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
-\Lambda_{c} \frac{\Delta u}{W_{c}} & 1 & 0 & 0 & -K_{c} F_{c} S_{c} \frac{\Delta u}{W_{c}} & 0 \\
0 & 0 & \cos (C \Delta u) & -\sin (C \Delta u) & 0 & 0 \\
0 & 0 & \sin (C \Delta u) & \cos (C \Delta u) & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\frac{7}{2} K_{c} S_{C} \frac{\Delta u}{W_{c}} & 0 & 0 & 0 & 2 K_{C} T_{c} \frac{\Delta u}{W_{c}} & 1
\end{array}\right]  \tag{12}\\
& \boldsymbol{\Psi}_{N C}\left(u_{c}, u_{0}\right)=\left[\begin{array}{ccc}
0 & \frac{2 \Delta u}{n_{c} a_{c} W_{c}} & 0 \\
-\frac{2 \Delta u}{n_{c} a_{c} W_{c}} & -\frac{\Lambda_{c} \Delta u^{2}}{n_{c} a_{c} W_{c}^{2}} & \frac{F_{c} K_{c} s_{c}\left(c_{u_{c}}-c_{u_{0}}+s_{u_{0}} \Delta u\right)}{n_{c} a_{c} W_{c}^{2}} \\
-\frac{c_{u_{c}}-c_{u_{c}}+c \Delta u}{n_{c} a_{c} \beta W_{c}} & 2 \frac{s_{u_{c}}-s_{u_{0}}+c \Delta u}{n_{c} a_{c} \beta W_{c}} & 0 \\
-\frac{s_{u_{c}}-s_{u_{0}+c \Delta u}}{n_{c} a_{c} \beta W_{c}} & -2 \frac{c_{u_{c}}-c_{u_{0}+C \Delta u}}{n_{c} a_{c} \beta W_{c}} & 0 \\
0 & 0 & \frac{s_{u_{c}-s_{u_{0}}}^{n_{c} a_{c} W_{c}}}{} \\
0 & \frac{7}{2} \frac{K_{c} s_{c} \Delta u^{2}}{n_{c} a_{c} W_{c}^{2}} & \binom{-\frac{\left(W_{c}+2 K_{c} T_{c}\right)\left(c_{u_{c}}-c_{u_{0}}\right)}{n_{c} a_{c} W_{c}^{2}}}{-\frac{2 K_{c} T_{c} s_{u_{0}} \Delta u}{n_{c} a_{c} W_{c}^{2}}}
\end{array}\right] \tag{13}
\end{align*}
$$

In Eqs. (12)-(13) $\Delta u$ indicates the variation of the mean argument of latitude of the chief orbit between the instant $t_{0}$ and $t$, i.e. $\Delta u=u_{c}-u_{0}$. The quantities $C$ and $\beta$ are constant coefficients that depend on the mean semi-major axis, eccentricity, and inclination of the chief orbit as follows

$$
\begin{equation*}
C=\frac{K_{c} Q_{c}}{W_{c}}, \quad \beta=1-C \tag{14}
\end{equation*}
$$

## Piecewise Constant Control Profile

In this study only the deputy is assumed to be maneuverable and capable of providing a thrust along $x$, $y$, and $z$ directions of its own Radial-Tangential-Normal (RTN) reference frame. This consists of a basis vectors with $x$ pointing radially away from the Earth to the deputy satellite, $z$ pointing along the direction of the angular momentum of the deputy orbit, and $y$ completing the right-handed ortho-normal basis.

The control acceleration profile is to be a piecewise constant function $\boldsymbol{F}\left(u_{c}\right)=\left[f_{x}\left(u_{c}\right), f_{y}\left(u_{c}\right), f_{z}\left(u_{c}\right)\right] \in$ $\mathbb{R}^{3}$ defined in the maneuvering interval $\left[u_{0}, u_{T}\right]\left(u_{T}=u_{c}(t=T)\right)$ as (see Figure 1)

$$
f_{(.)}\left(u_{c}\right)=\left\{\begin{array}{c}
f_{(.), j}=\text { const } \neq 0, u_{c(.), j, 0} \leq u_{c} \leq u_{c(.), j, f}, \quad j=1, \ldots, n  \tag{15}\\
0, \text { otherwise }
\end{array}\right.
$$

The term $n \in \mathbb{N}$ denotes the number of finite-time maneuvers within the interval $\left[u_{0}, u_{T}\right]$, whereas $u_{c(\cdot), j, 0}$ and $u_{c(\cdot), j, f}$ indicate the mean argument of latitude of the chief orbit at the beginning and the end of the $j$-th maneuver, respectively (or alternately the initial and final instant of time of the $j$-th maneuver according to the relationship reported in Eq. (10)).


Figure 1. Piecewise constant acceleration profile for a generic axis (.) of RTN reference frame.
By assuming that the relative motion is well described by the linear model (6)-(8) with the closed-form solution reported in Eqs. (11)-(13) and that the control acceleration has the form described in Eq. (15), the total change of the mean ROE through the maneuvering interval, $\Delta \delta \boldsymbol{\alpha}\left(u_{T}\right)$, can be analytically computed as follows ${ }^{11}$

$$
\Delta \delta \boldsymbol{\alpha}\left(u_{T}\right)=\sum_{j=1}^{n} \boldsymbol{\Phi}_{N C}\left(u_{T}, u_{c, j, f}\right) \boldsymbol{\Psi}_{N C}\left(u_{c, j, f}, u_{c, j, 0}\right)\left[\begin{array}{l}
f_{x, j}  \tag{16}\\
f_{y, j} \\
f_{z, j}
\end{array}\right], j=1, \ldots n
$$

being $\Delta \delta \boldsymbol{\alpha}\left(u_{T}\right)=\delta \boldsymbol{\alpha}\left(u_{T}\right)-\boldsymbol{\Phi}\left(u_{T}, u_{0}\right) \delta \boldsymbol{\alpha}_{0}$. As this study addresses the in-plane reconfiguration maneuver strategy, i.e. the control of the in-plane components of the mean ROE vector, namely $\delta a, \delta \lambda, \delta e_{x}$, and $\delta e_{y}$, the Eq. (16) can be written as

$$
\begin{gather*}
\sum_{j=1}^{n} \widetilde{U}_{y, j} f_{y, j}=\mu \Delta \delta a\left(u_{T}\right)  \tag{17}\\
\sum_{j=1}^{n}\left[\frac{\Lambda_{c}}{\beta W_{c}}\left(u_{T}-\widehat{U}_{j}\right) \widetilde{U}_{y, j} f_{y, j}+\widetilde{U}_{x, j} f_{x, j}\right]=-\mu \Delta \delta \lambda\left(u_{T}\right)  \tag{18}\\
\sum_{j=1}^{n}\left[\cos \left(\widehat{U}_{j}\right) \sin \left(\widetilde{U}_{y, j}\right) f_{y, j}+\frac{1}{2} \sin \left(\widehat{U}_{j}\right) \sin \left(\widetilde{U}_{x, j}\right) f_{x, j}\right]=\mu \Delta \delta e_{x}\left(u_{T}\right) \tag{19}
\end{gather*}
$$

$$
\begin{equation*}
\sum_{j=1}^{n}\left[\sin \left(\widehat{U}_{j}\right) \sin \left(\widetilde{U}_{y, j}\right) f_{y, j}-\frac{1}{2} \cos \left(\widehat{U}_{j}\right) \sin \left(\widetilde{U}_{x, j}\right) f_{x, j}\right]=\mu \Delta \delta e_{y}\left(u_{T}\right) \tag{20}
\end{equation*}
$$

where $\mu=\beta W_{c} n_{c} a_{c} / 4$. The terms $\widehat{U}_{j}, \widetilde{U}_{x, j}$, and $\widetilde{U}_{y, j}$ are related to the maneuver duration, $\tilde{u}_{(\cdot), j}=$ $\frac{u_{c(0, j, f}-u_{c(, j, 0}}{2}$, and location, $\hat{u}_{j}=\frac{u_{c(0, j, f}+u_{c(0, j, 0}}{2}$, by the following expressions

$$
\begin{equation*}
\widehat{U}_{j}=C u_{T}+\beta \widehat{u}_{j}, \quad \widetilde{U}_{x, j}=\beta \tilde{u}_{x, j}, \quad \widetilde{U}_{y, j}=\beta \tilde{u}_{y, j} \tag{21}
\end{equation*}
$$

From Eqs. (17)-(20) it is clear that the variation of the mean ROE at the end of the maneuvering interval, $\Delta \delta \check{\boldsymbol{\alpha}}\left(u_{T}\right)=\left[\Delta \delta a\left(u_{T}\right), \Delta \delta \lambda\left(u_{T}\right), \Delta \delta e_{x}\left(u_{T}\right), \Delta \delta e_{y}\left(u_{T}\right)\right]^{T}$, is a nonlinear function of the $j$-th maneuver' location and duration, $\hat{u}_{j}$ and $\tilde{u}_{(, j}$ respectively, while it depends linearly on the acceleration amplitudes, $f_{(0, j}$.

## Optimization Problem

The problem of designing the fuel-minimum in-plane reconfiguration maneuver can be formulated as a Lagrange optimal control problem with the performance index

$$
\begin{equation*}
J_{0}=\int_{u_{0}}^{u_{T}} \breve{\boldsymbol{F}}\left(u_{c}\right) d u_{c} \tag{22}
\end{equation*}
$$

where $\breve{\boldsymbol{F}}\left(u_{c}\right)=\left[f_{x}\left(u_{c}\right), f_{y}\left(u_{c}\right)\right]$. Hence, the optimization problem can be summarized as follows

$$
\begin{array}{lc} 
& \text { Find } \breve{\boldsymbol{F}}\left(u_{c}\right): u_{c} \rightarrow \mathcal{F} \subset \mathbb{R}^{2} \\
& \text { minimizing } J_{0} \\
& \text { subject to } \forall u_{c} \in\left[u_{0}, u_{T}\right]  \tag{23}\\
\text { dynamics constraint } \quad: \delta \dot{\widetilde{\boldsymbol{\alpha}}}\left(u_{c}\right)=\breve{\boldsymbol{A}}_{N C} \delta \breve{\boldsymbol{\alpha}}\left(u_{c}\right)+\breve{\boldsymbol{B}}_{N C}\left(u_{c}\right) \breve{\boldsymbol{F}}\left(u_{c}\right)^{T} \\
\text { boundary condition } \quad: \delta \breve{\boldsymbol{\alpha}}\left(u_{T}\right)=\delta \breve{\boldsymbol{\alpha}}_{\text {des }} \\
\text { control constraint } & :-f^{a d} \leq f_{(.)}\left(u_{c}\right) \leq f^{a d}, f^{a d}>0 \in \mathbb{R}
\end{array}
$$

The accent mark " " " indicates that only the in-plane components are taken into account. Then, the vector $\delta \breve{\boldsymbol{\alpha}}\left(u_{c}\right)$ is $\delta \breve{\boldsymbol{\alpha}}=\left[\delta a, \delta \lambda, \delta e_{x}, \delta e_{y}\right]^{T}, \breve{\boldsymbol{A}}_{N C}$ is a 4 x 4 matrix formed by taking the first four rows and columns of $\boldsymbol{A}_{N C}, \breve{\boldsymbol{B}}_{N C}$ is a $4 \times 2$ matrix formed by taking the first four rows and the two columns of $\boldsymbol{B}_{N C}$. Since the definition of the control profile given in Eq. (15), the optimal control problem (23) can straightforwardly transcribed into the following corresponding parameters optimization problem

$$
\begin{align*}
& \text { Find } \boldsymbol{x}=\left[f_{x, 1}, \ldots, f_{x, n}, f_{y, 1}, \ldots, f_{y, n}, \hat{u}_{1}, \ldots, \hat{u}_{n}, \tilde{u}_{x, 1}, \ldots, \tilde{u}_{y, n}, \tilde{u}_{x, 1}, \ldots, \tilde{u}_{y, n}\right]^{T} \\
& \text { minimizing } J^{*}=\frac{2}{W_{c}} \sum_{j=1}^{n}\left|f_{x, j}\right| \tilde{u}_{x, j}+\left|f_{y, j}\right| \tilde{u}_{y, j}, \quad j=1, \ldots n  \tag{24}\\
& \text { subject to } \forall \boldsymbol{x} \in \mathbf{X} \subset \mathbb{R}^{5 n} \\
& \text { boundary condition }: \Delta \delta \check{\boldsymbol{\alpha}}\left(u_{T}, \boldsymbol{x}\right)=\Delta \delta \breve{\boldsymbol{\alpha}}_{\text {des }} \\
& \text { parameter constraint } \quad: \boldsymbol{x}_{l b} \leq \boldsymbol{x} \leq \boldsymbol{x}_{u b}
\end{align*}
$$

where $\boldsymbol{x}_{l b} \in \mathbf{X}$ and $\boldsymbol{x}_{u b} \in \mathbf{X}$ denote the lower and upper boundaries of the parameter optimization vector, respectively. The ROE correction at the end of the maneuvering interval, $\Delta \delta \check{\boldsymbol{\alpha}}\left(u_{T}, \boldsymbol{x}\right)=$ [ $\left.\Delta \delta a, \Delta \delta \lambda, \Delta \delta e_{x}, \Delta \delta e_{y}\right]^{T}$, is computed by taking advantage of the closed-form solution of the linear dynamics system and is given by the expression (16) (or Eqs. (17)-(21)). The problem (24) is known as constrained nonlinear programming (NLP) problem.

## SOLUTION VIA MIXED-INTEGER LINEAR PROGRAMMING APPROACH

This section presents the Mixed-Integer Linear Programming (MILP) formulation derived to solve the optimization problem (23).

## MILP Description

The MILP is a special case of a Linear Programming (LP) problem (Reference 16) in which some variables are constrained to take only integer values. Constraints on such variables enable the inclusion of logical expressions in the optimization, encoding the combinatorial part of the problem ${ }^{9}$. As based on LP, all constraints as well as the objective function in MILP must be linear in the optimization parameters. A bunch of highly optimized commercial software exist for the solution of a MILP problem. In this study CPLEX software package (Reference 17) is used. It implements the branch-and-bound algorithm in conjunction with many adjustable heuristics, allowing quite large problems to be solved in practical computation times.

## MILP Formulation for Fuel-Minimum Reconfiguration Maneuver Design

The optimization problem described by Eq. (24) is nonlinear because of the final condition constraint and the definition of the objective function $J$. In further details, the variation of the mean ROE at the end of the maneuvering interval, $u_{T}$, is a nonlinear function of some optimization parameters, i.e. the maneuvers' locations and durations (see Eqs. (17)-(20)). Moreover, the objective function $J^{*}$ in Eq. (24) is a nonlinear function of the maneuvers' magnitudes, $f_{(.), j}$ with $j=1, \ldots, n$. Hence, to make the MILP approach suited for the determination of the fuel-minimum strategy for the formation reconfiguration, the constrained nonlinear programming problem (24) has to be first translated into a linear one. To this purpose, let subdivide the maneuvering interval $\left[u_{0}, u_{T}\right]$ in a finite number of sub-intervals, $N_{d}$, of length $\tilde{u}_{d, m}$ with $m=1, \ldots, N_{d}$, and associate to each of them a maneuver of magnitude $\boldsymbol{F}_{m}=\left[f_{x, m}, f_{y, m}\right]$, with $f_{x, m}, f_{y, m} \in\left[-f^{a d}, f^{a d}\right]$. In this way, the optimization parameters become the maneuvers' magnitudes related to the $m$-th sub-interval, $f_{x, m}$ and $f_{y, m}$ (see Figure 2). However, the discretization of the maneuvering interval does not solve the issue of nonlinearity of the objective function, which would be still nonlinear in $f_{x, m}$ and $f_{y, m}$. Then, let split $f_{(.), m}$ (with the subscript "(.)" indicating the directions $x$ and $y$ ) into two subsets, $f_{(.), m}^{+} \in\left[0, f^{a d}\right]$ and $f_{(.), m}^{-} \in\left[-f^{a d}, 0\right]$ such that the objective function in Eq. (24), $J^{*}$, can be rearranged as

$$
\begin{equation*}
J_{M I L P}^{*}=\sum_{m=1}^{N_{d}} \Delta v_{m}=\frac{2}{W_{c}} \sum_{m=1}^{N_{d}}\left[\left(f_{x, m}^{+}+f_{y, m}^{+}\right) \tilde{u}_{d, m}-\left(f_{x, m}^{-}+f_{y, m}^{-}\right) \tilde{u}_{d, m}\right] \tag{25}
\end{equation*}
$$

From Eq. (25), it is straightforward that the objective function $J_{M I L P}^{*}$ is now a linear function of the new set of optimization parameters, $\left[f_{(.), m}^{+}, f_{(.), m}^{-}\right]$.


Figure 2. Example of discretization for a generic axis (.) of RTN reference frame.
Finally, the MILP problem associated to the fuel-minimum reconfiguration strategy design can be formulated as

$$
\begin{gather*}
\text { minimizing } J_{M I L P}^{*}=\boldsymbol{c} \boldsymbol{x}_{M I L P}  \tag{26}\\
\text { subject to } \forall \boldsymbol{x}_{M I L P}=[\boldsymbol{\zeta}, \boldsymbol{\tau}]^{T}, \boldsymbol{\zeta} \in \boldsymbol{X} \subseteq \mathbb{R}^{4 N_{d}, \boldsymbol{\tau} \in \mathbf{Z}^{5 N_{d}}=\left\{\boldsymbol{\tau}: \tau_{s} \in \mathbb{N}, s=1, \ldots, 5 N_{d}\right\}} \\
\boldsymbol{H} \boldsymbol{x}_{M I L P} \leq \boldsymbol{Y}
\end{gather*}
$$

where

$$
\begin{equation*}
\boldsymbol{x}_{M I L P}=\left[\boldsymbol{F}_{x}^{+}, \boldsymbol{F}_{x}^{-}, \boldsymbol{\rho}_{x}^{+}, \boldsymbol{\rho}_{x}^{-}, \boldsymbol{F}_{y}^{+}, \boldsymbol{F}_{y}^{-}, \boldsymbol{\rho}_{y}^{+}, \boldsymbol{\rho}_{y}^{-}, \boldsymbol{\sigma}\right]^{T} \tag{27}
\end{equation*}
$$

being

$$
\begin{gather*}
\boldsymbol{F}_{(.)}^{+}=\left[\hat{f}_{(.), 1}^{+}, \ldots, \hat{f}_{(.), N_{d}}^{+}\right], \boldsymbol{F}_{(.)}^{-}=\left[\hat{f}_{(.), 1}^{-}, \ldots, \hat{f}_{(.), N_{d}}^{-}\right], \hat{f}_{(.), m}^{+}=\frac{f_{(\cdot), m}^{+}}{f^{r e f}} \in \mathbb{R}, \hat{f}_{(.), m}^{-}=\frac{f_{(.), m}^{-}}{f^{r e f}} \in \mathbb{R}  \tag{28}\\
\boldsymbol{\rho}_{(.)}^{+}=\left[p_{(.), 1}^{+}, \ldots, p_{(.), N_{d}}^{+}\right], \boldsymbol{\rho}_{(.)}^{-}=\left[p_{(.), 1}^{-}, \ldots, p_{(.), N_{d}}^{-}\right], \rho_{(.), m}^{+}, \rho_{(.), m}^{-} \in \mathbb{N}  \tag{29}\\
\boldsymbol{\sigma}=\left[\sigma_{1}, \ldots, \sigma_{N_{d}}\right], \sigma_{m} \in \mathbb{N} \tag{30}
\end{gather*}
$$

The subscript "(.)" stands for $x$ and $y$ in the above expressions. The variables $\boldsymbol{\rho}_{(.)}^{+}, \boldsymbol{\rho}_{(.)}^{-}$, and $\boldsymbol{\sigma}$ are the additional binary variables (integer variables all included in the range $[0,1]$ ) introduced to enforce different forms of constraints. These are expressed by a set of inequalities equations compactly written in the form $\boldsymbol{H} \boldsymbol{x}_{\text {MILP }} \leq \boldsymbol{Y}$ (see Eq. (26)). In the following section, more details on how to build the matrix $\boldsymbol{H}$ and the vector $\boldsymbol{Y}$ are given. In Eqs. (28)-(30) the term $f^{r e f}$ is a scale factor that allows limiting the variation of the maneuvers' magnitudes between 0 and 1, i.e. $f^{r e f}=f^{a d}$. This allows one to get quantities of the same order of magnitude in the optimization state vector, $\boldsymbol{x}_{\text {MILP }}$. In accordance to the definition of the MILP state vector, $\boldsymbol{x}_{\text {MILP }}$, reported in Eq. (27), the objective function can be rearranged as

$$
\begin{equation*}
J_{M I L P}^{*}=\boldsymbol{c} \boldsymbol{x}_{M I L P}=\frac{2 f^{r e f}}{W_{c}}\left[\widetilde{\boldsymbol{u}},-\widetilde{\boldsymbol{u}}, \mathbf{0}_{1 \times 2 N_{d}}, \widetilde{\boldsymbol{u}},-\widetilde{\boldsymbol{u}}, \mathbf{0}_{1 \times 3 N_{d}}\right] \boldsymbol{x}_{M I L P} \tag{31}
\end{equation*}
$$

where $\tilde{\boldsymbol{u}}=\left[\tilde{u}_{d, 1}, \ldots, \tilde{u}_{d, N_{d}}\right]$.
Maximum number of admissible maneuvers, $N_{I}$. In order to limit the number of maneuvers associated with the sub-intervals $N_{d}$, the binary variables $\boldsymbol{\rho}_{(.)}^{+}, \boldsymbol{\rho}_{(.)}^{-}, \boldsymbol{\sigma}$ are introduced. These variables are defined as

$$
\begin{gather*}
\rho_{(.), m}^{-}=\left\{\begin{array}{cc}
1, & \hat{f}_{(.), m}^{-}<0 \\
0, & \text { otherwise }
\end{array} \quad \rho_{(.), m}^{+}=\left\{\begin{array}{cc}
1, & \hat{f}_{(.), m}^{+}>0 \\
0, & \text { otherwise }
\end{array},\right.\right.  \tag{32}\\
\sigma_{m}= \begin{cases}1, & \rho_{x, m}^{+}+\rho_{x, m}^{-}+\rho_{y, m}^{+}+\rho_{y, m}^{-} \geq 1 \\
0, & \text { otherwise }\end{cases} \tag{33}
\end{gather*}
$$

In light of the above, the constraint on the maximum number of maneuvers can be expressed by the following $17 N_{d}+1$ inequalities,

$$
\begin{gather*}
0 \leq \hat{f}_{(.), m}^{+} \leq \rho_{(.), m}^{+}, \quad\left(1-\rho_{(.), m}^{-}\right) \geq \hat{f}_{(.), m}^{+} \geq 0, \quad m=1, \ldots, N_{d}  \tag{34}\\
-\rho_{(.), m}^{-} \leq \hat{f}_{(.), m}^{-} \leq 0, \quad-\left(1-\rho_{(.), m}^{+}\right) \leq \hat{f}_{(.), m}^{-} \leq 0  \tag{35}\\
M\left(\hat{f}_{(.), m}^{+}-\hat{f}_{(.), m}^{-}\right) \geq \rho_{(.), m}^{+}+\rho_{(.), m}^{-}, \quad M>0 \in \mathbb{R}  \tag{36}\\
\rho_{(.), m}^{+}+\rho_{(.), m}^{-} \leq \sigma_{m}, \quad-\left(\rho_{x, m}^{+}+\rho_{x, m}^{-}+\rho_{y, m}^{+}+\rho_{y, m}^{-}\right) \leq-\sigma_{m}, \quad \sum_{m=1}^{N_{d}} \sigma_{m} \leq N_{I} \tag{37}
\end{gather*}
$$

Note that $M$ is an arbitrary positive number whose value determines the value of the minimum admissible acceleration, i.e. $M=f^{r e f} / f_{\text {min }}$.

Final condition. At the end of the maneuvering time, $u_{T}$, the $r$-th mean ROE, $\delta \alpha_{r}\left(u_{T}\right)$, has to be equal to the desired corresponding mean ROE, $\delta \alpha_{r, \text { des }}$, i.e. $\delta \alpha_{r}\left(u_{T}\right)=\delta \alpha_{r, d e s}$ with $r=1, \ldots 4$. This equality constraints is transformed in an inequality constraint as follows

$$
\begin{equation*}
\left(\Delta \delta \alpha_{r}\left(u_{T}\right)-\Delta \delta \alpha_{r, \text { des }}\right) \leq \varepsilon_{r e l}\left|\Delta \delta \alpha_{r, \text { des }}\right|, \quad-\left(\Delta \delta \alpha_{r}\left(u_{T}\right)-\Delta \delta \alpha_{r, \text { des }}\right) \leq \varepsilon_{r e l}\left|\Delta \delta \alpha_{r, \text { des }}\right| \tag{38}
\end{equation*}
$$

where $\varepsilon_{r e l}$ is the user-defined tolerance. Defining the vectors $\widehat{\boldsymbol{U}}=\left[\widehat{U}_{1}, \ldots, \widehat{U}_{N_{d}}\right]$ and $\widetilde{\boldsymbol{U}}=\left[\widetilde{U}_{d, 1}, \ldots, \widetilde{U}_{d, N_{d}}\right]$, with $\widehat{U}_{m}=C u_{T}+\beta \hat{u}_{m}, \widetilde{U}_{d, m}=\beta \tilde{u}_{d, m}$ and $m=1, \ldots, N_{d}$, and $\boldsymbol{\mathcal { F }}=\left[\boldsymbol{F}_{x}^{+}, \boldsymbol{F}_{x}^{-}, \boldsymbol{F}_{y}^{+}, \boldsymbol{F}_{y}^{-}\right]$, Eq. (38) leads to the following eight inequalities (see Eqs. (17)-(21)),

$$
\begin{gather*}
\pm \frac{f^{r e f}}{\mu}\left[\mathbf{0}_{1 \times N_{d}}, \mathbf{0}_{1 \times N_{d}}, \widetilde{\boldsymbol{U}}, \widetilde{\boldsymbol{U}}\right] \boldsymbol{F}^{T} \leq \pm \Delta \delta a_{\text {des }}+\varepsilon_{\text {rel }}\left|\Delta \delta a_{\text {des }}\right|  \tag{39}\\
\pm \frac{f^{r e f}}{\mu}[-\widetilde{\boldsymbol{U}},-\widetilde{\boldsymbol{U}},-\mathbf{Y},-\mathbf{Y}] \mathcal{F}^{T} \leq \pm \Delta \delta \lambda_{\text {des }}+\varepsilon_{r e l}\left|\Delta \delta \lambda_{\text {des }}\right|  \tag{40}\\
\pm \frac{f^{r e f}}{\mu}\left[\frac{1}{2} \sin \widehat{\boldsymbol{U}}, \frac{1}{2} \sin \widehat{\boldsymbol{U}}, \cos \widehat{\boldsymbol{U}}, \cos \widehat{\boldsymbol{U}}\right] \mathcal{D}_{\widetilde{\boldsymbol{U}}} \boldsymbol{F}^{T} \leq \pm \Delta \delta \lambda_{\text {des }}+\varepsilon_{r e l}\left|\Delta \delta e_{x, \text { des }}\right|  \tag{41}\\
\pm \frac{f^{r e f}}{\mu}\left[-\frac{1}{2} \cos \widehat{\boldsymbol{U}},-\frac{1}{2} \cos \widehat{\boldsymbol{U}}, \sin \widehat{\boldsymbol{U}}, \sin \widehat{\boldsymbol{U}}\right] \mathcal{D}_{\widetilde{\boldsymbol{U}}} \boldsymbol{F}^{T} \leq \pm \Delta \delta \lambda_{\text {des }}+\varepsilon_{r e l}\left|\Delta \delta e_{y, \text { des }}\right| \tag{42}
\end{gather*}
$$

with

$$
\begin{gather*}
\boldsymbol{\Upsilon}=\frac{\Lambda_{c}}{\beta W_{c}}\left(u_{T} \mathbf{1}_{1 \mathrm{x} N_{d}}-\widehat{\boldsymbol{U}} \operatorname{diag}(\widetilde{\boldsymbol{U}})\right) \\
\mathcal{D}_{\widetilde{\boldsymbol{U}}}=\left[\begin{array}{cccc}
\operatorname{diag}(\sin (\widetilde{\boldsymbol{U}})) & \mathbf{0}_{N_{d^{\mathrm{x}}} N_{d}} & \mathbf{0}_{N_{d^{\mathrm{x}}} N_{d}} & \mathbf{0}_{N_{d^{\mathrm{x}} N_{d}}} \\
\mathbf{0}_{N_{d^{\mathrm{x}}} N_{d}} & \operatorname{diag}(\sin (\widetilde{\boldsymbol{U}})) & \mathbf{0}_{N_{d^{\mathrm{x}}} N_{d}} & \mathbf{0}_{N_{d^{\mathrm{x}} N_{d}}} \\
\mathbf{0}_{N_{d^{\mathrm{x}}} N_{d}} & \mathbf{0}_{N_{d^{\mathrm{x}}}} & \operatorname{diag}(\sin (\widetilde{\boldsymbol{U}})) & \mathbf{0}_{N_{d^{\mathrm{x}}}} \\
\mathbf{0}_{N_{d} \mathrm{x} N_{d}} & \mathbf{0}_{N_{d} \mathrm{x} N_{d}} & \mathbf{0}_{N_{d^{\mathrm{x}}} N_{d}} & \operatorname{diag}(\sin (\widetilde{\boldsymbol{U}}))
\end{array}\right] \tag{43}
\end{gather*}
$$

Notes on the discretization. In this study, a specific length of the sub-intervals is chosen in order to include in the feasible solution space the in-plane analytical solution reported in (Reference 11). In (Reference 11) it is shown that a 3 tangential maneuver strategy for formation reconfiguration can be can be analytically computed if the firings are separated from the reference angle $\bar{U}=\operatorname{atan}\left(\Delta \delta e_{y, d e s} / \Delta \delta e_{x, d e s}\right)$ by an angle equal to $k_{s, j} \pi$ with $k_{s, j} \in \mathbb{N}$ and $j=1,2,3$. In light of this, the maneuvering interval $\left[U_{0}, U_{T}\right]\left(U_{0}=C u_{T}+\beta u_{0}\right.$ and $U_{T}=C u_{T}+\beta u_{T}$ ) has to be discretized in such a way that the angular separation between the middle point of each sub-interval and the reference angle $\bar{U}$ is a multiple of $\pi$, i.e. $\widetilde{U}_{d}=\pi /(2 q)$ with $q \in \mathbb{N}$ (see Figure 3). For the sake of clarity, let us assume $\bar{U}$ is a real number greater than zero, $\bar{U}>0$. Then, the number of sub-intervals included in the $\left[\bar{U}, U_{T}\right]$ and $\left[U_{0}, \bar{U}\right]$ are

$$
\begin{equation*}
N_{R}=\text { floor }\left(\frac{\left(U_{T}-\left(\bar{U}+\widetilde{U}_{d}\right)\right)}{2 \widetilde{U}_{d}}\right) \quad \text { and } \quad N_{L}=\text { floor }\left(\frac{\left(\left(\bar{U}+\widetilde{U}_{d}\right)-U_{0}\right)}{2 \widetilde{U}_{d}}\right) \tag{44}
\end{equation*}
$$

respectively, where floor(.) is a function that rounds the element (.) to the nearest smaller integer. It is worth noting that if the quantities $\left(U_{T}-\left(\bar{U}+\widetilde{U}_{d}\right)\right) / 2 \widetilde{U}_{d}$ and $\left(\left(\bar{U}+\widetilde{U}_{d}\right)-U_{0}\right) / 2 \widetilde{U}_{d}$ are not integer numbers, the two sub-intervals in proximity of the boundaries of the interval $\left[U_{0}, U_{T}\right]$ have a length equal to

$$
\begin{equation*}
\widetilde{U}_{d, R}=\left(U_{T}-U_{R}\right) / 2 \quad \widetilde{U}_{d, L}=\left(U_{L}-U_{0}\right) / 2 \tag{45}
\end{equation*}
$$

where

$$
\begin{equation*}
U_{R}=\bar{U}+\widetilde{U}_{d}+N_{R} 2 \widetilde{U}_{d} \quad U_{L}=\bar{U}+\widetilde{U}_{d}-N_{L} 2 \widetilde{U}_{d} \tag{46}
\end{equation*}
$$

As a consequence, the total number of intervals is given by

$$
\begin{gather*}
N_{d}=N_{R}+N_{L}+\Delta_{R}+\Delta_{L} \\
\Delta_{R}=\left\{\begin{array}{ll}
1, & \widetilde{U}_{d, R} \neq 0 \\
0, & \widetilde{U}_{d, R}=0
\end{array} \quad \Delta_{L}= \begin{cases}1, & \widetilde{U}_{d, L} \neq 0 \\
0, & \widetilde{U}_{d, L}=0\end{cases} \right. \tag{47}
\end{gather*}
$$

Note that the interval $\left[U_{L}, U_{R}\right]$ is always uniformly subdivided. Then, the above time mesh can be defined "quasi-uniform" and becomes an uniform grid when $\widetilde{U}_{d, R}=\widetilde{U}_{d, L}=0$. The computation of number of subintervals for $\bar{U}<0$ is omitted here for brevity. However, a similar procedure can be used to determine $N_{d}$ when $\bar{U}<0$.


Figure 3. Discretization of the maneuvering interval for the optimal reconfiguration maneuver design through the MILP.

## SOLUTION VIA PARTICLE SWARM APPROACH

This section gives an overview of the details of the Particle Swarm Optimization (PSO) algorithm implemented for the solution of the optimization problem defined by Eq. (23).

## PSO Description

The PSO algorithm is a metaheuristic optimization method based on the cooperation between a fixed-size set (swarm) of $N_{S W}$ particles, i.e. a group of candidate solutions containing the optimization parameters ${ }^{8}$. The particles move through the set of acceptable and meaningful solutions, referred to as the Feasible Search Space (FSS), modifying their position, i.e. the values of the $\mathcal{M}$ optimization parameters associated with it, through an appropriate perturbation named velocity ${ }^{* *}$. During the evolution, the generic $i$-th particle is evaluated at the step $k$ through the performance index $J_{i}^{k}$, which takes into account the goal of the optimization and the imposed constraints. The evolution of a generic $i$-th particle can be computed by ${ }^{18}$

$$
\begin{equation*}
\boldsymbol{x}_{i}^{k+1}=\boldsymbol{x}_{i}^{k}+\boldsymbol{v}_{i}^{k+1} \quad k=1, \ldots, N_{i t e r} \tag{48}
\end{equation*}
$$

where $N_{\text {iter }}$ denotes the maximum number of iteration and $\boldsymbol{v}_{i}^{k+1}$ is the velocity term that, according to the unified version of PSO, is given by ${ }^{19}$

$$
\begin{equation*}
\boldsymbol{v}_{i}^{k+1}=w \boldsymbol{v}_{i}^{k}+u_{1} c_{p}\left(\boldsymbol{p}_{\text {best }, i}^{k}-\boldsymbol{x}_{i}^{k}\right)+u_{2} c_{l}\left(\boldsymbol{l}_{\text {best }, i}^{k}-\boldsymbol{x}_{i,}^{k}\right)+u_{3} c_{g}\left(\boldsymbol{g}_{\text {best }}^{k}-\boldsymbol{x}_{i}^{k}\right) \tag{49}
\end{equation*}
$$

The sum reported in Eq. (49) includes four terms. The first one is known as inertial component multiplied by a scaling factor $w$ and represents the vector pointing from $\boldsymbol{x}_{i}^{k-1}$ to $\boldsymbol{x}_{i}^{k}$. The second term is the cognitive

[^1]element and indicates the vector directed toward the personal best, $\boldsymbol{p}_{\text {best }, i}^{k}$, that is the particle with the performance index $J_{p, i}^{k}=\min _{1<\ell<k} J_{i}^{\ell}$. During the evolution, each particle remembers its previous personal best position $\boldsymbol{p}_{\text {best }, i}^{k}$ and always tends to return to that position. The third term is the local search vector, pointing toward the local best position, $\boldsymbol{l}_{\text {best, } i}^{k}$, i.e. the particle with the best objective function value in a small neighborhood of the $i$-th particle, $\mathcal{N}_{i}$, and then associated with the performance index $J_{l, i}^{k}=\min _{i \in \mathcal{N}_{i}} J_{p, i}^{k}$. Finally, the fourth term is the so-called social component and represents the vector directed toward the global best position, $\boldsymbol{g}_{b e s t}^{k}$, that is the particle with the performance index $J_{g}^{k}=\min _{1<i<N_{S W}} J_{p, i}^{k}$. The coefficients $u_{1}, u_{2}$ and $u_{3}$ are random numbers uniformly distributed in $[0,1]$. The quantities $c_{g}, c_{l}$, and $c_{p}$ are user-defined coefficients assumed to be constant in this study, whereas $w$ varies linearly along the optimization according with the following expression ${ }^{18}$
\[

$$
\begin{equation*}
w=w_{0}-\left(w_{0}-w_{f}\right) \frac{k-1}{N^{*}} \tag{50}
\end{equation*}
$$

\]

where $N^{*}$ is a user-defined parameter, $w_{0}$ and $w_{f}$ are the initial and final values, respectively. It is worth remarking that the particles can move only within the FSS since the velocity and the displacement are constrained to lie inside the hyper-parallelepiped with lower limits $\boldsymbol{x}_{\text {min }}$ and $\boldsymbol{v}_{\text {min }}$ and upper limits $\boldsymbol{x}_{\max }$ and $\boldsymbol{v}_{\text {max }}$, respectively ${ }^{18}$, i.e.

$$
\begin{gather*}
\boldsymbol{x}_{\min } \leq \boldsymbol{x}_{i}^{k} \leq \boldsymbol{x}_{\max } \quad \boldsymbol{v}_{\min } \leq \boldsymbol{v}_{i}^{k+1} \leq \boldsymbol{v}_{\max } \quad i=1, \ldots, N_{S W}  \tag{51}\\
\boldsymbol{v}_{\min }=-a\left(\boldsymbol{x}_{\max }-\boldsymbol{x}_{\min }\right) \quad \boldsymbol{v}_{\max }=a\left(\boldsymbol{x}_{\max }-\boldsymbol{x}_{\min }\right) \tag{52}
\end{gather*}
$$

where $a$ is a user-defined constant parameters. In order to make Eq. (51) satisfied, the position and velocity in Eqs. (48)-(49) are set equal to the lower (or upper) corresponding limits if $\boldsymbol{x}_{i}^{k}<\boldsymbol{x}_{\min }$ (or $\boldsymbol{x}_{i}^{k}>\boldsymbol{x}_{\max }$ ) and $\boldsymbol{v}_{i}^{k+1}<\boldsymbol{v}_{\text {min }}\left(\right.$ or $\left.\boldsymbol{v}_{i}^{k+1}>\boldsymbol{v}_{\max }\right)$, respectively, i.e.

$$
\left\{\begin{array}{r}
\boldsymbol{x}_{i}^{k}=\boldsymbol{x}_{\min }\left(\boldsymbol{x}_{i}^{k}=\boldsymbol{x}_{\max }\right) \wedge \boldsymbol{v}_{i}^{k+1}=\mathbf{0}_{1 \mathrm{x} \mathcal{M}}, \boldsymbol{x}_{i}^{k}<\boldsymbol{x}_{\min }\left(\boldsymbol{x}_{i}^{k}>\boldsymbol{x}_{\max }\right)  \tag{53}\\
\boldsymbol{v}_{i}^{k+1}=\boldsymbol{v}_{\min }\left(\boldsymbol{v}_{i}^{k+1}=\boldsymbol{v}_{\max }\right), \boldsymbol{v}_{i}^{k+1}<\boldsymbol{v}_{\min }\left(\boldsymbol{v}_{i}^{k+1}>\boldsymbol{v}_{\max }\right)
\end{array}\right.
$$

The equality constraints are treated by the PSO algorithm by converting them into inequalities and adding an additional term to the objective function $J$, i.e.

$$
\begin{equation*}
J=J_{0}+\sum_{r=1}^{N_{e}} d_{r} v_{r}\left(\boldsymbol{x}_{i}^{k}\right) \tag{54}
\end{equation*}
$$

where $N_{e}$ is the number of equality constraints, whereas $v_{r}\left(\boldsymbol{x}_{i}^{k}\right)$ is the penalty function associated to the equality constraints and defined as

$$
\begin{equation*}
v_{r}\left(\boldsymbol{x}_{i}^{k}\right)=\max \left(0, x_{r}\left(\boldsymbol{x}_{i}^{k}\right)-\Delta_{r}\right) \tag{55}
\end{equation*}
$$

$X_{r}\left(\boldsymbol{x}_{i}^{k}\right)$ represents the equality constraint function and $\Delta_{r}$ is a user-defined tolerance. It is noteworthy that the values of the weights $d_{r}$ must be carefully chosen and are problem dependent. Small values might imply excessive constraint violations, on the contrary high values of $d_{r}$ might render the problem ill-conditioned ${ }^{20}$. The PSO algorithm is terminated if the change of the best performance index is lower of a specific tolerance, $\varepsilon_{J}$, or the maximum number of iterations, $N_{\text {iter }}$, is achieved, i.e.

$$
\begin{equation*}
\delta J(\boldsymbol{x})=\frac{J^{k}-J^{k-1}}{J^{k-1}} \leq \varepsilon_{J} \quad \text { or } \quad k=N_{i t e r} \tag{56}
\end{equation*}
$$

## PSO Formulation for Fuel-Minimum Reconfiguration Maneuver Design

According to the constrained nonlinear optimization problem reported in Eq. (24), the $i$-th PSO particle is defined as follows

$$
\begin{equation*}
\boldsymbol{x}_{i, P S O}=\left[\boldsymbol{F}_{1}, \ldots, \boldsymbol{F}_{n}, \hat{u}_{1}, \ldots, \hat{u}_{n}, \widetilde{\boldsymbol{u}}_{1}, \ldots, \widetilde{\boldsymbol{u}}_{n}\right]^{T} \tag{57}
\end{equation*}
$$

being $n$ the number of maneuvers. The vectors $\boldsymbol{F}_{j} \in \mathbb{R}^{2}$ and $\widetilde{\boldsymbol{u}}_{j} \in \mathbb{R}^{2}$ with $j=1, \ldots, n$ are given by

$$
\begin{equation*}
\boldsymbol{F}_{j}=\left[f_{x, j}, f_{y, j}\right], \quad \tilde{\boldsymbol{u}}_{j}=\left[\tilde{u}_{x, j}, \tilde{u}_{y, j}\right] . \tag{58}
\end{equation*}
$$

The performance index $J^{*}$ in Eq. (24) is modified by adding a penalty function associated to the final condition constraints, i.e.

$$
\begin{gather*}
J_{P S O}=J^{*}+\sum_{r=1}^{4} d_{r} v_{r}  \tag{59}\\
v_{r}=\max \left(0,\left|\frac{\Delta \delta \alpha_{r}\left(\boldsymbol{x}_{i, P S O}\right)^{-\Delta \delta \alpha_{r, d e s}}}{\Delta \delta \alpha_{r, d e s}}\right|-\Delta_{r}\right)
\end{gather*}
$$

where the constraint tolerance, $\Delta_{r}$, is evaluated with the adaptive decreasing tolerance technique described in (Reference 21). In order to obtain a solution compatible with the maneuvering time interval $\left[u_{0}, u_{T}\right]$ and the maximum available acceleration, $f^{a d}$, the following constraints have to be imposed

$$
\begin{gather*}
\hat{u}_{1}-\tilde{u}_{(.), 1}>0 \\
\left(\hat{u}_{h}-\hat{u}_{h-1}\right)-\left(\tilde{u}_{(.) h}-\tilde{u}_{(.), h-1}\right)>0, \quad h=2, \ldots, n-1 \\
u_{T}-\hat{u}_{n}-\tilde{u}_{(.), n}>0  \tag{60}\\
-f^{a d} \leq f_{(.), j} \leq f^{a d}, \quad j=1, \ldots, n
\end{gather*}
$$

Since the above constraints are linear relationships between the PSO optimization parameters, they do not need to be included in the extended cost function but rather they can be treated as a specific form of Eq. (51), being

$$
\begin{align*}
& \boldsymbol{x}_{\text {min }}=\left[\boldsymbol{F}_{1}^{\text {min }}, \ldots, \boldsymbol{F}_{n}^{\min }, \hat{u}_{1}^{\min }, \ldots, \hat{u}_{n}^{\min }, \widetilde{\boldsymbol{u}}_{1}^{\min }, \ldots, \widetilde{\boldsymbol{u}}_{n}^{\min }\right]^{T}  \tag{61}\\
& \boldsymbol{F}_{j}^{\text {min }}=\left[-f^{a d},-f^{a d}\right], \quad \widetilde{\boldsymbol{u}}_{j}^{\text {min }}=\left[\tilde{u}_{\text {min }}, \tilde{u}_{\text {min }}\right], \quad \widehat{u}_{j}^{\text {min }}=u_{0}  \tag{62}\\
& \boldsymbol{x}_{\max }=\left[\boldsymbol{F}_{1}^{\max }, \ldots, \boldsymbol{F}_{n}^{\max }, \hat{u}_{1}^{\max }, \ldots, \hat{u}_{n}^{\max }, \widetilde{\boldsymbol{u}}_{1}^{\max }, \ldots, \widetilde{\boldsymbol{u}}_{n}^{\max }\right]^{T}  \tag{63}\\
& \boldsymbol{F}_{j}^{\max }=\left[f^{a d}, f^{a d}\right], \quad \widetilde{\boldsymbol{u}}_{j}^{\max }=\left[\tilde{u}_{\max }, \tilde{u}_{\max }\right], \quad \hat{u}_{j}^{\max }=u_{T}  \tag{64}\\
& \boldsymbol{v}_{\text {max }}=\left[\boldsymbol{v}_{\text {max }}^{F}, \boldsymbol{v}_{\text {max }}^{\widehat{u}}, \boldsymbol{v}_{\text {max }}^{\widetilde{u}}\right]^{T}  \tag{65}\\
& \boldsymbol{v}_{\max }^{F}=a_{F}\left[\left(\boldsymbol{F}_{1}^{\max }-\boldsymbol{F}_{1}^{\min }\right), \ldots,\left(\boldsymbol{F}_{n}^{\max }-\boldsymbol{F}_{n}^{\min }\right)\right]  \tag{66}\\
& \boldsymbol{v}_{\text {max }}^{\widehat{u}}=a_{\widehat{u}}\left[\left(\hat{u}_{1}^{\max }-\hat{u}_{1}^{\min }\right), \ldots,\left(\hat{u}_{n}^{\max }-\hat{u}_{n}^{\min }\right)\right]  \tag{67}\\
& \boldsymbol{v}_{\max }^{\widetilde{u}}=a_{\widetilde{u}}\left[\left(\widetilde{\boldsymbol{u}}_{1}^{\max }-\widetilde{\boldsymbol{u}}_{1}^{\min }\right), \ldots,\left(\widetilde{\boldsymbol{u}}_{n}^{\max }-\widetilde{\boldsymbol{u}}_{n}^{\min }\right)\right] \tag{68}
\end{align*}
$$

The quantities $a_{F}, a_{\widehat{u}}$, and $a_{\tilde{u}}$ are user-defined constant parameters. Finally, let us remind that $\boldsymbol{v}_{\text {min }}=$ $-\boldsymbol{v}_{\text {max }}$.

## NUMERICAL SIMULATIONS

This section presents the trajectories designed using the proposed approaches, namely the PSO and the MILP, pointing out their performances in terms of maneuver cost and accuracy. A numerical satellite orbit simulator including the Earth's oblateness effect is exploited to propagate the initial states of deputy and chief expressed in the Earth Centered Inertial (ECI) reference frame (J200). The control acceleration profile obtained in the deputy RTN reference frame is projected in ECI and added as external accelerations to the deputy's motion. Note that the linear mapping developed by Brouwer and Lyddane (Reference 22,23) is used to transform the mean orbital elements to osculating and vice versa. As illustrated in Figure 4, following a chain of transformations comprising the nonlinear relations between Cartesian ECI state and osculating orbital elements and the aforementioned linear map to convert the osculating to mean elements, the distance between the current relative orbit and the desired one can be computed as, i.e.

$$
\begin{equation*}
\epsilon_{\Delta \delta \alpha_{r}}(t)=\frac{\Delta \delta \alpha_{r}^{\text {num }}(t)-\Delta \delta \alpha_{r, \text { des }}}{\left|\Delta \delta \alpha_{r, d e s}\right|} \quad r=1, \ldots, 4 \tag{69}
\end{equation*}
$$



Figure 4. Numerical simulations layout.
The initial chief mean orbit and the relative orbit used in the numerical simulations are listed in Table 1 and Table 2 (first row), respectively. Table 2 (second row) also reports the desired mean relative orbit at the end of maneuvering time. The chief moves on a circular orbit with an altitude of 200 km . The reconfiguration maneuver lasts 6 chief orbital periods, i.e. $u_{T}=12 \pi$ (rad) corresponding to $T=528.6(\mathrm{~min})$, with the initial mean argument of latitude equal to zero, i.e. $u_{0}=0(\mathrm{rad})$. According to Eq. (16), the values of $a_{c} \delta \boldsymbol{\alpha}_{0}$ and $a_{c} \delta \boldsymbol{\alpha}_{d e s}$ lead to $a_{c} \Delta \delta \boldsymbol{\alpha}_{d e s}=a_{c}\left(\delta \boldsymbol{\alpha}_{d e s}-\boldsymbol{\Phi}\left(u_{T}, u_{0}\right) \delta \boldsymbol{\alpha}_{\mathbf{0}}\right)=[-0.03,2.2,0.0394,0.11968]^{T}(\mathrm{~km})$.

## Table 1. Initial chief mean orbit.

| $a_{c}(\mathrm{~km})$ | $e_{c}(\mathrm{dim})$ | $i_{c}(\mathrm{deg})$ | $\omega_{c}(\mathrm{deg})$ | $\Omega_{c}(\mathrm{deg})$ | $M_{c, 0}(\mathrm{deg})$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 6578 | 0 | 8 | 0 | 0 | 0 |

Table 2. Initial and desired mean relative orbits.

|  | $a_{c} \delta a(\mathrm{~m})$ | $a_{c} \delta \lambda(\mathrm{~m})$ | $a_{c} \delta e_{x}(\mathrm{~m})$ | $a_{c} \delta e_{y}(\mathrm{~m})$ |
| :---: | :---: | :---: | :---: | :---: |
| Initial relative orbit | 30 | $-11 \times 10^{3}$ | 0 | -50 |
| Desired relative orbit | 0 | $-10.5 \times 10^{3}$ | 45 | 70 |

All simulations, including the computation of PSO and MILP solutions, are obtained using a personal computer with an Intel® Core ${ }^{\text {TM }} \mathrm{i} 7-2677 \mathrm{M}$ CPU @ 1.8 GHz processor and 4 GB of RAM.

## Fuel-minimum Reconfiguration Maneuver Strategy via MILP

In this section the performances of MILP approach are analyzed. Let us recall that the MILP problem is solved using the CPLEX software package ${ }^{17}$.

Fuel-minimum maneuver. Here, the trajectory and the control profile obtained by the MILP formulation are presented. The "quasi-uniform" time mesh discussed above is used to discretize the maneuvering interval with $2 \widetilde{U}_{d}=0.7854 \mathrm{rad}$ (i.e. $q=4$ ), corresponding to a number of sub-intervals $N_{d}=48$. In other words, the duration of each maneuver is imposed to be at least $\Delta t=11.045 \mathrm{~min}$. The values of boundary subintervals $\widetilde{U}_{d, L}$ and $\widetilde{U}_{d, R}$ are 0.374 rad and 0.3554 rad , respectively (see Figure 3). The maximum admissible control acceleration is set to $f^{a d}=3 \times 10^{-4} \mathrm{~m} / \mathrm{s}^{2}$.

Figure 5 (left) illustrates the acceleration control profile along the $y$ direction of the RTN reference frame. It consists of three tangential maneuvers located at $\widehat{U}_{1}=1.2526 \mathrm{rad}$ (or $\widehat{u}_{1}=1.1442 \mathrm{rad}$ ), $\widehat{U}_{2}=4.3942 \mathrm{rad}$ (or $\hat{u}_{2}=4.2951 \mathrm{rad}$ ), and $\widehat{U}_{3}=32.6686 \mathrm{rad}\left(\right.$ or $\hat{u}_{3}=32.6536 \mathrm{rad}$ ), with $f_{y, 1}=0.268 \times 10^{-4} \mathrm{~m} / \mathrm{s}^{2}, f_{y, 2}=$ $-0.7112 \times 10^{-4} \mathrm{~m} / \mathrm{s}^{2}$, and $f_{y, 3}=0.175 \times 10^{-4} \mathrm{~m} / \mathrm{s}^{2}$. The control acceleration along the radial direction of the local reference frame is null. This result is to be expected due to the higher efficiency of the along-track maneuvers with respect to the radial ones ${ }^{10}$. The angular separation between the maneuvers' mid-points and the reference angle $\bar{U}=1.2526 \mathrm{rad}$ is an integer multiple of $\pi$, i.e. $k_{s, 1}=\left(\widehat{U}_{1}-\bar{U}\right) / \pi=0, k_{s, 2}=$ $\left(\widehat{U}_{2}-\bar{U}\right) / \pi=1$, and $k_{s, 2}=\left(\widehat{U}_{3}-\bar{U}\right) / \pi=10$. The total maneuver cost for the MILP reconfiguration strategy is $0.0765 \mathrm{~m} / \mathrm{s}$. The tolerance on the final condition constraint is set equal to $\varepsilon_{r e l}=1 \times 10^{-11}$. In addition, CPLEX provides the solution to the MILP problem in 0.033 s .


Figure 5. Acceleration profile (left) and accuracy (right) given by the MILP approach.

The same figure (right) shows the mean ROE state variation over time scaled by the final desired ROE correction (see Eq. (69)). At the end of the maneuvering interval, the final desired position is achieved with the value of accuracies, $\epsilon_{\Delta \delta \alpha_{r}}(t)$ with $r=1, \ldots 4$, listed in Table 3. Accordingly, the total accuracy defined as

$$
\begin{equation*}
\epsilon_{T o t}=\sqrt{\left(\epsilon_{\Delta \delta a}(T)\right)^{2}+\left(\epsilon_{\Delta \delta \lambda}(T)\right)^{2}+\left(\epsilon_{\Delta \delta e_{x}}(T)\right)^{2}+\left(\epsilon_{\Delta \delta e_{y}}(T)\right)^{2}} \tag{70}
\end{equation*}
$$

is equal to $5.09 \times 10^{-3} \mathrm{~mm}$.
Table 3. In-plane accuracies given by the MILP approach.

| $\left\|\epsilon_{\Delta \delta a}(T)\right\| a_{c}(\mathrm{~m})$ | $\left\|\epsilon_{\Delta \delta \lambda}(T)\right\| a_{c}(\mathrm{~m})$ | $\left\|\epsilon_{\Delta \delta e_{x}}(T)\right\| a_{c}(\mathrm{~m})$ | $\left\|\epsilon_{\Delta \delta e_{y}}(T)\right\| a_{c}(\mathrm{~m})$ | $\epsilon_{T o t} a_{c}(\mathrm{~m})$ |
| :---: | :---: | :---: | :---: | :---: |
| $1.512 \times 10^{-3}$ | $1.129 \times 10^{-3}$ | $4.718 \times 10^{-3}$ | $0.377 \times 10^{-3}$ | $5.09 \times 10^{-3}$ |

Ultimately, Figure 6 illustrates the evolution of the relative position projected on the along-track/crosstrack plane of the RTN reference frame. In the figure also the three firing intervals are depicted (see magenta, green and cyan markers corresponding to the first, the second and the third maneuver). The initial and the desired relative positions are indicated by the red and yellow markers, respectively.


Figure 6. In-plane trajectory obtained through the MILP approach.

Analysis of discretization. This section aims at investigating the effects of the maneuvering interval discretization on the reconfiguration strategy performances. Here the "quasi-uniform" time mesh is considered, with the parameter $q$ ranging from 1 to 60 , i.e. with $N_{d}$ varying between 13 and 719 and, then, with $\Delta t$ varying between 44.18 min and 0.73 min . Figure 7 shows the total maneuvering cost (left) and the average acceleration scaled by $f^{a d}=3 \times 10^{-4} \mathrm{~m} / \mathrm{s}^{2}$ (right) over the number of sub-intervals, $N_{d}$. Accordingly, while the total maneuvering cost, $\Delta v_{\text {Tot }}$, decreases with the increase of sub-intervals, the mean of absolute values of maneuvers' amplitude, i.e. $\bar{f}_{y}=\sum_{j}^{n}\left|f_{y, j}\right| / n$, rises (the control acceleration along the radial direction of the RTN reference frame is not illustrated because it is always null). In other words, the MILP solution tends to the optimal impulsive one, as illustrated by the left plot in Figure 7 where the $\Delta v_{T o t}=0.074562 \mathrm{~m} / \mathrm{s}$ associated with the optimal impulsive solution is depicted by the black line (the reader is addressed to Reference 15 for the details on the computation of the optimal impulsive solution in $J_{2}$ perturbed orbits). The result reported in the same plot shows that seven maneuvers with a duration of 0.73 min (i.e. $q=60$ ) should be performed over the maneuvering interval to obtain the minimum achievable $\Delta v_{\text {Tot }}$ of $7.4569 \times 10^{-2} \mathrm{~m} / \mathrm{s}$. From the conducted analysis it turns out that a minimum number of three maneuvers are needed to meet the constraints in Eq. (24). Moreover, for the specific initial and final conditions considered in this analysis, all three maneuver strategies provide an angular separation between the maneuvers' mid-points and the reference angle $\bar{U}=1.2526 \mathrm{rad}$ equal to $k_{s, j} \pi$ with $k_{s, j} \in \mathbb{N}$ and $j=1,2,3$, i.e. $\left(\widehat{U}_{j}-\bar{U}\right)=k_{s, j} \pi$. This implies that the class of analytical solution derived by the authors in (Reference 11) is a sub-optimal solution for the fuel-minimum reconfiguration maneuvering problem. The best fuel-minimum 3-maneuver strategy in terms of maneuvering cost requires a total delta-V of $7.4602 \times 10^{-2} \mathrm{~m} / \mathrm{s}$ and it is obtained with a value of $q=27$. In addition, a $36.33 \%$ improvement of delta- V is noted with respect to the reference delta- $\mathrm{V}, \Delta v_{T o t}$ at $N_{d}=$ 13. Furthermore, when $q \geq 9$ (i.e., $N_{d} \geq 121$ ) the change of $\Delta v_{\text {Tot }}$ is less than $0.5 \%$. Ultimately, the MILP never provides the extremal control solution, i.e. the absolute value of at least one of the maneuvers' amplitudes is lower than $f^{a d}$ (see Figure 7 (right)).

Figure 8 shows the relationship between the computing time required to obtain the MILP solution and the discretization of the maneuvering time. As expected, the computing time increases with the number of subintervals, $N_{d}$. In fact, the dimension of the optimization problem linearly grows with the parameter, $N_{d}$; the optimizer state dimension is $9 N_{d}$ (see Eq. (27)) whereas the number of inequality constraints associated to the MILP optimal problem is $17 N_{d}+9$.


Figure 7. Maneuver cost (left) and mean scaled acceleration (right) over the number of sub-intervals.


Figure 8. Computing time over the number of sub-intervals.

## Fuel-minimum Reconfiguration Maneuver Strategy via PSO

In this section the performances of PSO approach are presented. Let us remark that the PSO algorithm has been implemented by the authors using Matlab. However, the developed software has not been optimized. Future work will include the improvement of the computational performance of the presented PSO algorithm.

Fuel-minimum maneuver. Here, the relative trajectory and the control profile given by the PSO approach are showed. The PSO parameters used for the numerical simulations are summarized in Table 4.

Table 4. PSO parameters.

| $N_{S W}$ | $c_{p}$ | $c_{l}$ | $c_{g}$ | $N^{*}$ | $N_{\text {iter }}$ | $w^{\dagger \dagger}$ | $\varepsilon_{J}$ | $\Delta_{r}^{\dagger \dagger}$ | $a_{F}$ | $a_{\widehat{u}}$ | $a_{\widetilde{u}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 200 | 1.5 | 2 | 0.5 | 1500 | 10000 | $0.4 \rightarrow 1.1$ | $1 \times 10^{-10}$ | $0.5 \rightarrow 1 \times 10^{-11}$ | 0.01 | 0.01 | 0.01 |

[^2]In order to make the PSO results comparable with those obtained by the MILP approach (i.e. $q=4$ and, accordingly, $N_{d}=48$ ), three radial/tangential maneuvers are imposed (i.e., $n=3$ ) with a minimum and maximum durations of $2 \tilde{u}_{\min }=2 \tilde{u}_{\max }=0.787 \mathrm{rad}$ (i.e. $\Delta t=11.045 \mathrm{~min}$ ). Hence, the PSO particle $\boldsymbol{x}_{i, P S O}$ in Eq. (57) contains 18 optimization parameters. The maximum admissible control acceleration is set to $f^{a d}=3 \times 10^{-4} \mathrm{~m} / \mathrm{s}^{2}$.

Figure 9 (left) shows the control solution given by the PSO algorithm. It consists of three tangential maneuvers placed at $\widehat{U}_{1}=4.398 \mathrm{rad}\left(\right.$ or $\left.\widehat{u}_{1}=4.299 \mathrm{rad}\right), \widehat{U}_{2}=23.243 \mathrm{rad}\left(\right.$ or $\left.\widehat{u}_{2}=23.200 \mathrm{rad}\right)$, and $\widehat{U}_{3}=$ $26.381 \mathrm{rad}\left(\right.$ or $\hat{u}_{3}=26.347 \mathrm{rad}$ ), with $f_{y, 1}=-0.412 \times 10^{-4} \mathrm{~m} / \mathrm{s}^{2}, f_{y, 2}=-0.299 \times 10^{-4} \mathrm{~m} / \mathrm{s}^{2}$, and $f_{y, 3}=$ $0.443 \times 10^{-4} \mathrm{~m} / \mathrm{s}$. It is worth noting that also the PSO nullifies the radial maneuvers, providing an alongtrack maneuvering strategy for the reconfiguration of the satellite formation. The total cost of the reconfiguration maneuver is $7.6512 \times 10^{-2} \mathrm{~m} / \mathrm{s}$. In the same figure (right) the variation of mean ROE over the maneuvering interval is depicted. As showed by the accuracies' values reported in Table 5, the PSO provides a total final relative error of $4.76 \times 10^{-3} \mathrm{~m}$. The implemented PSO algorithm provides the solution to the constrained nonlinear programming problem (24) in 14.8 min , converging to the near-optimal solution in about 5000 iterations.

Table 5. In-plane accuracies given by the PSO approach.

| $\left\|\epsilon_{\Delta \delta a}(T)\right\| a_{c}(\mathrm{~m})$ | $\left\|\epsilon_{\Delta \delta \lambda}(T)\right\| a_{c}(\mathrm{~m})$ | $\left\|\epsilon_{\Delta \delta e_{x}}(T)\right\| a_{c}(\mathrm{~m})$ | $\left\|\epsilon_{\Delta \delta e_{y}}(T)\right\| a_{c}(\mathrm{~m})$ | $\epsilon_{T o t} a_{c}(\mathrm{~m})$ |
| :---: | :---: | :---: | :---: | :---: |
| $1.509 \times 10^{-3}$ | $1.128 \times 10^{-3}$ | $4.362 \times 10^{-3}$ | $0.411 \times 10^{-3}$ | $4.76 \times 10^{-3}$ |

Finally, Figure 10 illustrated the projection of the relative orbit on the along-track/cross-track plane of the RTN reference frame. In the figure, the location of the maneuvers along the trajectory is also depicted as well as the initial and final positions.


Figure 9. Acceleration profile (left) and accuracy (right) given by the PSO approach.


Figure 10. In-plane trajectory obtained through the PSO approach.

## MILP/PSO Formulation Difference

The proposed approaches for the solution of the reconfiguration problem present some substantial differences. First, while the PSO algorithm can be directly used to solve the constrained nonlinear programming problem (24), the MILP approach requires the i) discretization of the maneuvering interval to eliminate the nonlinearities related to the boundary constraints (see Eq. (17)-(20)) and ii) the introduction of an additional set of optimization parameters (i.e. $f_{(.)}^{+}$and $\left.f_{(.)}^{-}\right)$to make the objective function linear. The discretization has the main disadvantage of reducing the search domain of the optimal control solution. In fact, the maneuvers are forced to be located at specific instants and last a specific interval of time depending on the length of subintervals defined by the user. On the other hand, the discretization procedure allows one to include the number of maneuvers in the optimization problem. On the contrary, the presented PSO approach involves only continuous variables that lie in the FSS. However, it requires the user to define the number of maneuvers, reducing the degree of freedom associated with the optimization process.

Finally, it is worth pointing out that the MILP formulation involves much larger number of optimization parameters than the PSO to solve the optimal control problem associate with the design of the fuel-minimum maneuvering strategy. However, this does not jeopardize the computational performance, which remains higher than the PSO one by several order of magnitudes as showed by the results presented above.

## CONCLUSION

This paper addressed the design of the fuel-minimum maneuvering strategy for the spacecraft formation reconfiguration in $J_{2}$-perturbed near-circular orbit. The reconfiguration problem has been formulated as an optimal control problem, assuming that the maneuverable spacecraft can perform only a series of constant finite-time maneuvers to control the relative in-plane configuration, i.e. the control acceleration profile is assumed to be a piecewise constant function over the maneuvering interval. Two different methods have been proposed in this work to solve the aforementioned optimal control problem, namely the Mixed-Integer Linear Programming approach and the Particle Swarm Optimization. The former requires the discretization of the maneuvering interval to eliminate the nonlinearities related to the boundary constraints as well as the introduction of an additional set of optimization parameters to make the objective function linear, whereas the latter can directly solve the constrained nonlinear programming problem associated to the design of the fueloptimal reconfiguration strategy. Linear dynamics model based on relative orbit element parameterization and its associated closed-form solution is used to impose the boundary conditions, avoiding the dynamics integration within the optimization process.

Simulation results demonstrated the effectiveness of the proposed methodologies. When the control solutions given by the two aforementioned approaches are added to the deputy nonlinear dynamics, the final formation configuration is achieved with an accuracy of the order of millimeter. Moreover, the analyses carried out showed the computational efficiency of the MILP approach against the PSO one. Even when a fine discretization is used, in fact, the computation of the fuel-optimal control solution takes less than 10 seconds, making it suited for the on-board implementation.

Possible future works include the improvement of the dynamics model used for the derivation of the fueloptimal strategy by taking into account the effects of atmospheric drag and solar radiation pressure, and the inclusion of path constraints in the formulation of optimization problem to prevent satellite collision during the maneuver.

## ACKNOWLEDGMENTS

The authors would like to thank the United States Air Force Research Laboratory, Space Vehicles Directorate, for sponsoring this investigation under contract FAA9453-16-C-0029.

## APPENDIX A: CONTROL INFLUENCE MATRIX

The elements of the control influence matrix $\boldsymbol{\Gamma}_{F}$ (see Eq. (4))

$$
\begin{align*}
& \gamma_{13}=\gamma_{51}=\gamma_{52}=\gamma_{61}=\gamma_{62}=0 \\
& \gamma_{11}=\frac{2 e_{d} s_{f_{d}}}{n_{d} \eta_{d} a_{c}}, \quad \gamma_{12}=\frac{2\left(1+e_{d} c_{f_{d}}\right)}{n_{d} \eta_{d} a_{c}} \\
& \gamma_{21}=-\frac{\eta_{d} e_{d} c_{f_{d}}}{a_{d} n_{d}\left(1+\eta_{d}\right)}-\frac{2 \eta_{d}^{2}}{a_{d} n_{d}\left(1+e_{d} c_{f_{d}}\right)} \\
& \gamma_{22}=-\frac{\eta_{d} e_{d}\left[\left(2+e_{d} c_{f_{d}}\right) s_{f_{d}}\right]}{a_{d} n_{d}\left(1+\eta_{d}\right)\left(1+e_{d} c_{f_{d}}\right)}, \quad \gamma_{23}=-\frac{\eta s_{\theta_{d}}\left(c_{i_{c}}-c_{i_{d}}\right)}{a_{d} n_{d}\left(1+e_{d} c_{f_{d}}\right) s_{i}}  \tag{71}\\
& \gamma_{31}=\frac{\eta_{d} s_{\theta_{d}}}{a_{d} n_{d}}, \quad \gamma_{32}=\frac{\eta_{d}\left(2+e_{d} c_{f_{d}}\right) c_{\theta_{d}}+\eta_{d} e_{x, d}}{a_{d} n_{d}\left(1+e_{d} c_{f_{d}}\right)} \\
& \gamma_{33}=\frac{\eta_{d} e_{y, d} s_{\theta_{d}} \operatorname{cotg}\left(i_{d}\right)}{a_{d} n_{d}\left(1+e_{d} c_{f_{d}}\right)}, \quad \gamma_{41}=-\frac{\eta_{d} c_{\theta_{d}}}{a_{d} n_{d}} \\
& \gamma_{42}=\frac{\eta_{d}\left(2+e_{d} c_{f_{d}}\right) s_{\theta_{d}}+\eta_{d} e_{y, d}}{a_{d} n_{d}\left(1+e_{d} c_{f_{d}}\right)}, \quad \gamma_{43}=-\frac{\eta_{d} e_{x, d} s_{\theta_{d}} \operatorname{cotg}\left(i_{d}\right)}{a_{d} n_{d}\left(1+e_{d} c_{f_{d}}\right)} \\
& \gamma_{53}=\frac{\eta_{d} s_{\theta_{d}}}{a_{d} n_{d}\left(1+e_{d} c_{f_{d}}\right)}, \quad \gamma_{63}=\frac{\eta_{d} c_{\theta_{d} s_{c}}}{a_{d} n_{d}\left(1+e_{d} c_{f}\right) s_{s_{d}}} .
\end{align*}
$$

## REFERENCES

${ }^{1}$ G. Gaias and S. D'Amico, "Impulsive Maneuvers for Formation Reconfiguration Using Relative Orbital Elements." Journal of Guidance, Control, and Dynamics, Vol. 38, No. 6, 2015, DOI: 10.2514/1.G000189.
${ }^{2}$ S.S. Vaddi, K.T. Alfriend, S. R. Vadali, and P. Sengputa, "Formation establishment and reconfiguration using impulsive control." Journal of Guidance, Control, and Dynamics, Vol. 28, No. 2, 2005, pp. 262-268, DOI: 10.2514/1.6687.
${ }^{3}$ M. Lawn, G. Di Mauro, and R. Bevilacqua, "Guidance Solutions for Spacecraft Planar Rephasing and Rendezvous Using Input Shaping." Journal of Guidance, Control, and Dynamics, 2017, DOI: 10.2514/1.G002910.
${ }^{4}$ A. Richards, T. Schouwenaars, J.P. How, and E. Feron, "Spacecraft Trajectory Planning with Avoidance Constraints Using Mixed-integer Linear Programming." Journal of Guidance, Control, and Dynamics, Vol. 25, No. 4, pp. 755-764, DOI:10.2514/2.4943.
${ }^{5}$ B. Acikmese, D. Scharf, F. Hadaegh, and E. Murray, "A Convex Guidance Algorithm for Formation Reconfiguration." AIAA Guidance, Navigation, and Control Conference and Exhibit, Keystone (CO, USA), 2006, DOI:10.2514/6.20066070.
${ }^{6}$ G. T. Huntington, A. V. Rao, "Optimal Reconfiguration of Spacecraft Formations Using the Gauss Pseudospectral method." Journal of Guidance, Control, and Dynamics, Vol. 31, No. 3, pp. 689-698, DOI:10.2514/1.31083.
${ }^{7}$ M. Mauro, F. Bernelli-Zazzera, "Optimization of Low-thrust Trajectories for Formation Flying with Parallel Multiple Shooting." AIAA/AAS Astrodynamics Specialist Conference and Exhibit, Keystone (CO, USA), 2006. DOI: 10.2514/6.2006-6747.
${ }^{8}$ K. E. Pasopoulos, M. N. Vrahatis, "Particle Swarm Optimization and Intelligence: Advances and Applications." Information Science Reference, Hershey (PA, USA), chaps. 2-4, 2010.
${ }^{9}$ C. A. Floudas, "Nonlinear and Mixed-Integer Optimization: Fundamentals and Applications." Oxford University Press, New York (NY, USA), chaps. 1995.
${ }^{10}$ S. D'Amico, "Autonomous Formation Flying in Low Earth Orbit." Ph.D. thesis, Delft University of Technology, Delft, Netherlands (2010).
${ }^{11}$ G. Di Mauro, D. Spiller, R. Bevilacqua, J. Sullivan, and S. D’Amico, "Continuous Maneuvers of Spacecraft Formation Flying Reconfiguration Using Relative Orbit Elements." 9 ${ }^{\text {th }}$ International Workshop of Spacecraft Formation Flying, Boulder (CO, USA), 2017.
${ }^{12}$ K. Alfriend, S. Vadali, P. Gurfil, J. How, and L. Breger, "Spacecraft Formation Flying: Dynamics, Control, and Navigation." Elsevier Astrodynamics Series, 2010.
${ }^{13}$ H. Schaub, J. Junkins, "Analytical Mechanics of Space Systems." AIAA education series, American Institute of Aeronautics and Astronautics, 2003.
${ }^{14}$ Roscoe, C.W. T., Westphal, J.J., Griesbach, J.D., and Schaub H., "Formation Establishment and Reconfiguration Using Differential Elements in J2-Perturbed Orbits." Journal of Guidance, Control, and Dynamics, Vol. 38, pp. 1725-1740, 2015, DOI: 10.2514/1.G000999.
${ }^{15}$ M. Chernick, S. D'Amico, "New Closed-form Solutions for Optimal Impulsive Control of Spacecraft Relative Motion," AIAA/AAS Astrodynamics Specialist Conference, 2016. DOI:10.2514/6.2016-5659.
${ }^{16}$ D. G. Luenberger, Y. Ye, "Linear and Nonlinear Programming.", Springer, New York (NY, USA), Chap. 2, 2008.
${ }^{17}$ ILOG AMPL CPLEX System Version 7.0 User's Guide, ILOG, Inc., Incline Village, NV, 2000, pp. 17-53,
${ }^{18}$ D. Spiller, L. Ansalone, and F. Curti, "Particle Swarm Optimization for Time-optimal Spacecraft Reorientation with Keep-out cones." Journal of Guidance, Control, and Dynamics, Vol. 39, No. 2, pp. 312-325, 2016. DOI: 10.2514/1.G001228.
${ }^{19}$ K. Parsopoulos and M. Vrahatis, "Parameter Selection and Adaptation in Unified Particle Swarm Optimization." Mathematical and Computer Modelling, Vol. 46, pp. 198-213, 2007. DOI: 10.1016/j.mcm.2006.12.019
${ }^{20}$ M. Pontani, B. A. Conway, "Optimal Finite-Thrust Rendezvous Trajectories Found via Particle Swarm Algorithm." Journal of Spacecraft and Rockets, Vol. 50, No. 6, pp. 1222-1234, 2013, DOI: 10.2514/1.A32402.
${ }^{21}$ D. Spiller, F. Curti, C. Circi, "Minimum-time Reconfiguration Maneuvers of Satellite Formations Using Perturbation Forces, Journal of Guidance, Control, and Dynamics, Vol. 40, No. 5, pp. 1130-1143, 2017, DOI:10.2514/1.G002382.
${ }^{22}$ D. Brouwer, "Solution of the Problem of Artificial Satellite Theory without Drag," Astronautical Journal, Vol. 64, pp. 378-397, 1963. DOI: 10.1086/107958.
${ }^{23}$ R. Lyddane, "Small Eccentricities or Inclinations in the Brouwer Theory of the Artificial Satellite," Astronautical Journal, Vol. 68, pp. 555-558., 1963. DOI: 10.1086/109179.


[^0]:    * PostDoc Associate, Department of Mechanical and Aerospace Engineering, ADAMUS Laboratory, University of Florida, 939 Sweetwater Dr., Gainesville, FL 326116250.
    ${ }^{\dagger}$ PhD Candidate, Department of Aerospace and Mechanical Engineering, Sapienza University of Rome, via Eudossiana 18, 00100.
    *) Associate Professor, Department of Mechanical and Aerospace Engineering, ADAMUS Laboratory, University of Florida, 939 Sweetwater Dr., Gainesville, FL 326116250.
    ${ }^{\text {§ }}$ Associate Professor, School of Aerospace Engineering, Sapienza University of Rome, via Salaria 851, 00100.

[^1]:    **Note that the terms position and velocity are referred to the search space of the optimization parameters and do not have any physical meaning.

[^2]:    ${ }^{\dagger}$ The first term indicates the value of the parameter at the beginning of the evolution whereas the second one denotes the final value of the parameter.

