# RENDEZVOUS VIA DIFFERENTIAL DRAG WITH UNCERTAINTIES IN THE DRAG MODEL 

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#### Abstract

At Low Earth Orbits a differential in the drag acceleration between coplanar spacecraft can be used for controlling their relative motion in the orbital plane. Current methods for determining the drag acceleration may result in errors due to the inaccuracy of density models and misrepresentation of the drag coefficient. In this work a novel methodology for relative maneuvering of spacecraft under bounded uncertainties in the drag acceleration is developed. In order to vary the relative drag acceleration, the satellites modify their pitch angle. Two approaches are proposed. First, a dynamical model composed of the mean semi-major axis and argument of latitude is utilized for describing long range maneuvers. For this model, a Linear Quadratic Regulator (LQR) is implemented, accounting for the uncertainties in the drag force. This controller guarantees asymptotic stability of the system up to a certain magnitude of the state vector, which is determined by the uncertainties. Furthermore, based on a cartesian relative motion formulation, a min-max control law is designed for short range maneuvers. This provides asymptotic stability under bounded uncertainties. The two approaches are tested in numerical simulations illustrating a long range re-phasing, performed using the LQR controller, followed by a short range rendezvous maneuver, accomplished using the min-max controller.


## INTRODUCTION

The inertial acceleration $\mathrm{a}_{\mathrm{D}}$ generated by the drag force is usually modeled as [1, Page 549]

$$
\begin{equation*}
\mathbf{a}_{\mathrm{D}}=-\rho \mathbf{v}_{\text {rel }}\left\|\mathbf{v}_{\text {rel }}\right\| C_{B} \tag{1}
\end{equation*}
$$

where $\mathbf{v}_{\text {rel }}$ denotes the velocity vector of the satellite relative to the atmosphere, $\rho$ represents the atmospheric density, and $C_{B}$ is the ballistic coefficient. The ballistic coefficient is defined as $C_{B} \triangleq$ $C_{D} S /(2 m)$, where $S$ stands for the cross-sectional area, $C_{D}$ is the drag coefficient, and $m$ denotes the mass of the spacecraft. Equation (1) shows that $\mathrm{a}_{\mathrm{D}}$ acts always in the direction opposed to the vector $\mathbf{v}_{\text {rel }}$. Since the atmosphere inertial velocity is usually a small component compared to the inertial velocity of a LEO satellite, ${ }^{2}$ it is usually neglected, leading to use the inertial velocity of the satellite $\mathbf{v}$ instead of $\mathbf{v}_{\text {rel }}$ in Eq. (1). Under the assumed model, drag forces cannot have components perpendicular to the instantaneous plane of motion. This is a significant limitation for the use of drag to maneuver. Yet, within the plane of motion, certain maneuvers can be achieved by use of drag only, reducing the propellant needs in certain missions. ${ }^{3,4}$

[^0]The main effect of the drag force is reducing the semi-major axis of the orbit. The rate of the semimajor axis decay is basically determined by quantities involved in Eq. (1). If these quantities are judiciously exploited, relative accelerations between two or more satellite can be generated, such that they are steered towards relative states desirable for specific multiple-satellite applications. This idea is usually termed differential-drag (DD) maneuvering. In recent years, the use of DD for satellite relative maneuvers has been actively investigated due to its potential for reducing the propellant needs in formation flying and cluster flight missions.

Leonard et. al. ${ }^{5}$ derived a control scheme that utilizes drag plates acting at either maximum or minimum drag. To that end, the in-plane dynamics was modeled with the Clohessy-Wiltshire equations, ${ }^{6}$ and the density was assumed constant. Carter and Humi ${ }^{7}$ derived linearized equations of relative motion that include effects caused by drag, assuming a drag force model proportional to the square of the velocity. Kumar and $\mathrm{Ng}^{4}$ extended the work by Leonard et. al. to consider other acting perturbations, erroneous measurements and inter-satellite distances slightly larger than those considered by Leonard et. al., but still in the order of magnitude of few tens of kilometers.

Bevilacqua et. al. ${ }^{8,9}$ utilized the linear formulation of Schweighart and Sedwick ${ }^{10}$ to derive a DD-based controller for steering the in-plane relative coordinates to zero. They assumed a constant density, and actuation provided by drag plates generating either minimum or maximum drag force. Based on Schweighart-Sedwick equations ${ }^{10}$ and the same aforementioned drag-plates actuation, Pérez and Bevilacqua ${ }^{11}$ removed the assumption on constant atmospheric density and proposed an adaptive controller to perform rendezvous. Ben-Yaacov and Gurfil ${ }^{12}$ used DD to perform relative maneuvers for cluster-keeping purposes. Dell'Elce and Kerschen ${ }^{13}$ utilized pseudospectral methods and model predictive control for planning and effectuating rendezvous maneuvers.

So far, the research in this area has been mainly oriented to close-proximity maneuvers, whereas the potential of differential-drag based maneuvers can go beyond close-proximity operations. Indeed, the ORBCOMM constellation ${ }^{14}$ utilizes differential drag, in an open-loop manner, to control the relative phase angles of the satellites. One could also envision applications that require to guide satellites, which are initially separated by large distances (order of magnitude of $\sim 1000 \mathrm{~km}$.) in the same orbital plane, along trajectories that drive them into close-proximity configurations. For this purpose, DD can be also utilized, enabling reductions in the propellant requirements.

One of the main difficulties of designing differential-drag maneuvers is the inherent uncertainties existing in some of the quantities in Eq. (1). The models of the Earth's atmospheric density field, as well as the drag coefficient values associated to various satellite geometries can be inaccurate, ${ }^{15}$ leading to uncertainties in the effects of differential-drag based maneuvers.

One of the goals of this work is to design differential-drag cooperative maneuvers explicitly considering uncertainties in the drag models, which at the best knowledge of the authors has not received much attention. These maneuvers are aimed at steering the satellites from given initial conditions, to close-proximity configurations oriented to a rendezvous.

Two insights are proposed. The first insight formulates the problem using a linearized relative motion representation, based on orbital elements. In order to perform the linearization, the main assumption is that the difference in mean semi-major axes between the two satellites, is small compared to the mean semi-major axes values. Unlike other linearized formulations, ${ }^{6,10}$ this model allows large distances between the satellites, as long as both mean semi-major axes are kept close one to each other. In this manner, one can consider initially large phase-separations between the satellites, and drive them into close-proximity configurations. Under this model, an LQR controller
is proposed, where the input of the system is determined by the cross-sectional area of the satellites, i.e. the pitch angles. Moreover, an analysis of the convergence of the system driven by the proposed LQR controller in the presence of bounded uncertainties is presented. This leads to determine gains ensuring that the system, even under these uncertainties, still convergences up to a certain norm of the state vector. Since the cross-sectional area of the satellites is limited, an assessment of the system under saturation is provided, showing that convergence is still achieved.

A second novel method is presented, which utilizes a min-max control ${ }^{16}$ approach on a linearized cartesian formulation. ${ }^{10}$ This enables to drive the in-plane states of the satellites from closeproximity configurations towards rendezvous conditions. The min-max method allows to explicitly deal with bounded uncertainties in the drag acceleration, including the atmospheric density field and the ballistic coefficients.

## DESCRIPTION OF THE PROBLEM

Let two satellites, Chaser and Target, be in coplanar circular orbits. The main goal of this work is to derive DD based, closed-loop controllers that steer the satellites to an encounter. These maneuvers will be performed by varying the drag force generated on either satellite, with no thrust usage.


Figure 1. Assumed geometry of the satellite.

Motivated by the rapid increase in the number of missions composed of cube-sats, this work assumes that the satellite geometries are rectangular parallelepipeds, as the one illustrated in the 3D View of Figure 1. In this paper, these bodies are endowed with one rotational degree of freedom, being the axis of rotation always perpendicular to the plane of motion and depicted as a "dash-dot" line in the 3D View of Fig. 1. The input considered for the control laws is the attitude of the satellite parametrized by the angle $\beta$, according to Fig. 1. To measure $\beta$, define a line lying on the orbital plane, perpendicular to the velocity vector, as the dashed line illustrated in the 2D View and Orbital View of Fig. 1. $\beta$ is measured from the aforementioned line towards the velocity vector. In Fig. 1, $S_{A}$ and $S_{B}$ denote the surfaces perpendicular to the plane of motion. The total cross-sectional area will be given by

$$
\begin{equation*}
S=S_{A}|\cos \beta|+S_{B}|\sin \beta| \tag{2}
\end{equation*}
$$

Note the vector $\mathbf{v}$ indicates the inertial velocity of the satellite. Changing $\beta$ modifies the crosssectional area $S$ and consequently the magnitude of the exerted acceleration $a_{\mathrm{D}}$. Due to the periodicity of $S(\beta)$, for the purposes of this work, $\beta$ can be restricted to the range $\beta \in\left[0^{\circ}, 90^{\circ}\right]$, which allows to remove the absolute value bars from Eq. (2). Then, Equation (2) can be reformulated as

$$
\begin{equation*}
S=S_{0} \cos (\beta-\psi) \tag{3}
\end{equation*}
$$

where $S_{0}=\sqrt{S_{A}^{2}+S_{B}^{2}}$ and $\psi=\operatorname{atan} 2\left(\frac{S_{B}}{S_{0}}, \frac{S_{A}}{S_{0}}\right)$. It can be seen that $S$ has a maximum at $\beta=\psi>0$, and a minimum at $\beta=90^{\circ}$. Hence, for the purposes of this work, the range of $\beta$ can be restricted even more: $\beta \in\left[\psi, 90^{\circ}\right]$.

## ADDRESSING THE PROBLEM WITH ORBITAL ELEMENTS

This section presents an approach to drive two satellites that are initially in circular orbits, in the same orbital plane but separated in phase, i.e. with different arguments of latitude, towards a closeproximity configuration. This configuration is attained by matching the mean semi-major axes $\bar{a}$ of the satellites and the mean argument of latitude $\bar{\theta}$. Notice that for initially circular orbits (very low eccentricities), neither the Earth's oblateness nor drag effects increase the mean eccentricities, i.e. the orbits remain circular* . Hence, matching $\bar{a}$ and $\bar{\theta}$ brings the two satellites into a close-proximity configuration.

## Dynamic Model

Let $\bar{a}, \bar{e}, \bar{i}, \bar{\omega}$, and $\bar{M}$ respectively denote the secular (or mean) components of the semi-major axis, eccentricity, inclination, argument of perigee, and mean anomaly. Under the influence of drag and the first term of the gravitational geopotential due to zonal harmonics $\left(J_{2}\right)$, the time-variation of the mean argument of perigee $\bar{\omega}$ and the mean mean anomaly $\bar{M}$ are respectively given by Mishne ${ }^{17}$ as:

$$
\begin{equation*}
\dot{\bar{\omega}}=\frac{3}{4} J_{2} \bar{n}\left(\frac{R_{e q}}{\bar{p}}\right)^{2}\left(5 \cos ^{2} \bar{i}-1\right) \quad \dot{\bar{M}}=\bar{n}+\frac{3}{4} J_{2} \bar{n}\left(\frac{R_{e q}}{\bar{p}}\right)^{2} \sqrt{1-\bar{e}^{2}}\left(3 \cos ^{2} \bar{i}-1\right) \tag{4}
\end{equation*}
$$

where $\bar{p}$ and $\bar{n}$ denote the parameter (semilatus rectum) of the orbits and the mean motion respectively, and $R_{e q}$ represents the mean equatorial radius of the Earth.

The argument of latitude $\theta$ is defined as $\theta \triangleq \omega+f$, where $f$ denotes the true anomaly. Assuming circular orbits, the variation of the mean argument of latitude can be modeled as

$$
\begin{equation*}
\dot{\bar{\theta}}=\dot{\bar{M}}+\dot{\bar{\omega}}=\sqrt{\frac{\mu}{\bar{a}^{3}}}+\frac{3}{4} J_{2} \sqrt{\frac{\mu}{\bar{a}^{3}}}\left(\frac{R_{e q}}{\bar{a}}\right)^{2}\left(8 \cos ^{2} \bar{i}-2\right) \tag{5}
\end{equation*}
$$

On the other hand, the rate of change of the mean semi-major axis can be formulated using the Gauss variational equations (GVE) and the premise that the effects on the mean elements due to disturbances other than $J_{2}$, can be approximated by the effects of the same disturbances on the respective osculating elements ${ }^{\dagger}$. Using the GVE resolved in tangential and normal axes [21, Page 489], the time-variation of the semi-major axis $a$ is formulated as

$$
\begin{equation*}
\dot{a}=\frac{2 a^{2} v}{\mu} \Gamma_{t} \tag{6}
\end{equation*}
$$

where $v \triangleq\|\mathbf{v}\|$, and $\Gamma_{t}$ represents the disturbance acceleration components along the inertial velocity vector. Since the perturbation due to $J_{2}$ has no effect on $\bar{a}, \Gamma_{t}$ accounts for accelerations due to drag only. Hence,

$$
\begin{equation*}
\Gamma_{t}=-\frac{1}{2} \rho v^{2} \frac{C_{D}}{m} S_{0} \cos (\beta-\psi) \tag{7}
\end{equation*}
$$

[^1]Recalling that for circular orbits $v=\sqrt{\mu / a}, \dot{\bar{a}}$ is approximated by

$$
\begin{equation*}
\dot{\bar{a}}=-2 \sqrt{\mu \bar{a}} \rho C_{B 0} \cos (\beta-\psi) \tag{8}
\end{equation*}
$$

where $C_{B 0}=C_{D} S_{0} /(2 m)$. Equations (5) and (8) constitute the dynamic model that will be used throughout this section. Recall that under $J_{2}$ and drag influence, the mean inclination $\bar{i}$ remains constant, and thus it actually represents a parameter. It is important to mention that the reduced state $[\bar{\theta}, \bar{a}]^{\top}$ represents only the in-plane motion.

Considering two satellites of equal geometry and size, Chaser and Target, the differential elements are defined as:

$$
\begin{align*}
& \Delta \bar{\theta} \triangleq \bar{\theta}_{C}-\bar{\theta}_{T}  \tag{9}\\
& \Delta \bar{a} \triangleq \bar{a}_{C}-\bar{a}_{T} \tag{10}
\end{align*}
$$

In Eqs. (9) and (10), as well as for the remainder of the paper, the sub-indices $(\cdot)_{C}$ or $(\cdot)_{T}$ refer to the parameter or variable $(\cdot)$ associated to the Chaser or Target respectively. The lack of sub-index in certain expressions indicates that the expression is valid for either spacecraft indistinctly.

From Eqs. (5) and (8), the evolution of $\Delta \bar{\theta}$ and $\Delta \bar{a}$ are given by

$$
\begin{gather*}
\Delta \dot{\bar{\theta}}=\sqrt{\mu}\left(\frac{1}{\bar{a}_{T}^{\frac{3}{2}}}\left[\frac{1}{\left(1+\frac{\Delta \bar{a}}{\bar{a}_{T}}\right)^{\frac{3}{2}}}-1\right]+\frac{3}{4} J_{2} R_{e q}^{2}\left(8 \cos ^{2} \bar{i}-2\right) \frac{1}{\bar{a}_{T}^{\frac{7}{2}}}\left[\frac{1}{\left(1+\frac{\Delta \bar{a}}{\bar{a}_{T}}\right)^{\frac{7}{2}}}-1\right]\right)  \tag{11}\\
\Delta \dot{\bar{a}}=-2 \sqrt{\mu}\left(\rho_{C} \sqrt{\bar{a}_{T}} \sqrt{1+\frac{\Delta \bar{a}}{\bar{a}_{T}}} u\left(\beta_{C}\right)-\rho_{T} \sqrt{\bar{a}_{T}} u\left(\beta_{T}\right)\right) \tag{12}
\end{gather*}
$$

where

$$
\begin{equation*}
u\left(\beta_{C}\right) \triangleq C_{B 0} \cos \left(\beta_{C}-\psi\right) \quad u\left(\beta_{T}\right) \triangleq C_{B 0} \cos \left(\beta_{T}-\psi\right) \tag{13}
\end{equation*}
$$

For brevity, in the forthcoming developments, $u\left(\beta_{C}\right)$ and $u\left(\beta_{T}\right)$ will be denoted by $u_{C}$ and $u_{T}$, respectively.

Notice that, for given densities $\rho_{C}$ and $\rho_{T}$, the nonlinear equations (8) and (5) depend solely on $\bar{a}$, but not on $\bar{\theta}$. Hence, the dynamic equations (11) and (12) can be linearized about $\Delta \bar{a} / \bar{a}_{T}=0$. Generally speaking, in multiple-satellite missions that involve coordinated relative motion, the mean semi-major axes should be similar; otherwise there are high natural drift rates that can rapidly dismantle any desired configuration. Hence, the linearization assumes that $\frac{\left|\bar{a}_{C}-\bar{a}_{T}\right|}{\bar{a}_{T}} \ll 1$. Yet, notice that the linear system is still valid for large differences in the argument of latitude, and hence possibly large distances. The linearization of the right hand side (RHS) of Eq. (11) yields

$$
\begin{equation*}
\Delta \dot{\bar{\theta}}=-P_{0} \Delta \bar{a} \tag{14}
\end{equation*}
$$

where

$$
\begin{equation*}
P_{0} \triangleq \sqrt{\mu}\left[\frac{3}{2} \frac{1}{\bar{a}_{T}^{5 / 2}}+\frac{21}{8} \frac{J_{2} R_{e q}^{2}\left(8 \cos ^{2} \bar{i}-2\right)}{\bar{a}_{T}^{9 / 2}}\right] \tag{15}
\end{equation*}
$$

On the other hand, to avoid terms containing products of the input and the state, the term $\sqrt{1+\frac{\Delta \bar{a}}{\bar{a}_{T}}} u\left(\beta_{C}\right)$ in Eq. (12) is approximated by

$$
\begin{equation*}
\sqrt{1+\frac{\Delta \bar{a}}{\bar{a}_{T}}} u\left(\beta_{C}\right) \simeq u\left(\beta_{C}\right)-u\left(\beta_{C}\right) \frac{1}{2} \frac{\Delta \bar{a}}{\bar{a}_{T}} \simeq u\left(\beta_{C}\right) \tag{16}
\end{equation*}
$$

(i.e. it is only the zero ${ }^{\text {th }}$ order term of the corresponding Taylor series), leading to

$$
\begin{equation*}
\Delta \dot{\bar{a}}=-2 \sqrt{\mu} \sqrt{\bar{a}_{T}}\left(\rho_{C} u_{C}-\rho_{T} u_{T}\right) \tag{17}
\end{equation*}
$$

Defining $\mathbf{w} \triangleq[\Delta \bar{\theta}, \Delta \bar{a}]^{\top}$, the obtained linear system can be written as follows

$$
\dot{\mathbf{w}}=\left[\begin{array}{cc}
0 & -P_{0}  \tag{18}\\
0 & 0
\end{array}\right] \mathbf{w}+\left[\begin{array}{l}
0 \\
b
\end{array}\right]\left(-\rho_{C} u_{C}+\rho_{T} u_{T}\right)
$$

where $b \triangleq 2 \sqrt{\mu \bar{a}_{T}}$.
Assuming that over a maneuver, $\bar{a}_{T}$ does not vary significantly (i.e.
$\left(\bar{a}_{T}\left(t_{0}\right)-\bar{a}_{T}\left(t_{f}\right)\right) / \bar{a}_{T}\left(t_{0}\right) \ll 1$ where $t_{0}$ and $t_{f}$ denote the initial and final time of the maneuver respectively), $\bar{a}_{T}$ can be substituted by $\bar{a}_{T}\left(t_{0}\right)$ and the dynamical system (18) may be considered linear time-invariant (LTI). Numerical simulations will validate the use of the obtained linearized model. The input of this system is given by the term $-\rho_{C} u_{C}+\rho_{T} u_{T}$. Notice that the uncertainties in the densities $\rho_{C}$ and $\rho_{T}$, and possibly in the drag coefficients $C_{D}$, constitute uncertainties in the input.

## Controller Derivation

Assuming no constraints on the input, an infinite-horizon LQR approach will be initially implemented. Since the range of $u \triangleq-\rho_{C} u_{C}+\rho_{T} u_{T}$ is limited, a saturation function will be proposed in the sequel, and it will be shown that convergence is still achieved. Recall that, in an infinite horizon LQR, if $\mathbf{w}$ denotes the state vector, the input is given by [22, Chapter 3.3]

$$
\begin{equation*}
\mathbf{u}_{L Q R}=-R^{-1} \mathbf{B}^{\top} \mathbf{P} \mathbf{w} \tag{19}
\end{equation*}
$$

where $\mathbf{P}$ is the matrix that solves the algebraic Riccati Equation. For the problem in question, the matrices are given by

$$
\mathbf{A}=\left[\begin{array}{cc}
0 & -P_{0}  \tag{20}\\
0 & 0
\end{array}\right] \quad \mathbf{B}=\left[\begin{array}{l}
0 \\
b
\end{array}\right] \quad R>0 \quad \mathbf{O} \triangleq\left[\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right]
$$

and

$$
\mathbf{Q} \triangleq\left[\begin{array}{cc}
q_{1} & 0  \tag{21}\\
0 & q_{2}
\end{array}\right] \text {, with } q_{1}>0 \text { and } q_{2}>0 \quad \mathbf{P} \triangleq\left[\begin{array}{ll}
\Pi_{1} & \Pi_{2} \\
\Pi_{2} & \Pi_{3}
\end{array}\right] \text {, positve definite }
$$

For the discussed problem, $\mathbf{P}$ is sought such that the matrix

$$
\begin{equation*}
\mathbf{A}^{*} \triangleq \mathbf{A}-\mathbf{B} R^{-1} \mathbf{B}^{\top} \mathbf{P} \tag{22}
\end{equation*}
$$

be Hurwitz, i.e. all the eigenvalues of $\mathbf{A}^{*}$ must have negative real part. Hence, the expressions for the entries of the matrix $\mathbf{P}$ are determined as

$$
\begin{equation*}
\Pi_{1}=\frac{\sqrt{q_{1}} \sqrt{2 P_{0} \sqrt{q_{1} R}+q_{2} b}}{P_{0} \sqrt{b}} \quad \Pi_{2}=-\frac{\sqrt{q_{1} R}}{b} \quad \Pi_{3}=\frac{\sqrt{R} \sqrt{2 P_{0} \sqrt{q_{1} R}+q_{2} b}}{b^{3 / 2}} \tag{23}
\end{equation*}
$$

and the gain $\mathbf{K}$ is

$$
\begin{equation*}
\mathbf{K}=R^{-1} \mathbf{B}^{\top} \mathbf{P}=\left[-\sqrt{\frac{q_{1}}{R}} \quad \sqrt{2 \frac{P_{0}}{b} \sqrt{\frac{q_{1}}{R}}+\frac{q_{2}}{b}}\right] \tag{24}
\end{equation*}
$$

With $\mathbf{K}$ already computed, then the desired input to the system is

$$
\begin{equation*}
u_{\mathrm{des}}=-\mathbf{K}[\Delta \bar{\theta}, \Delta \bar{a}]^{\top} \tag{25}
\end{equation*}
$$

Now, assuming that

$$
\begin{equation*}
\left(-\rho_{C}+\rho_{T} \zeta\right) C_{B 0} \leq u_{\mathrm{des}} \leq\left(-\rho_{C} \zeta+\rho_{T}\right) C_{B 0} \tag{26}
\end{equation*}
$$

where $\zeta \triangleq \cos (\pi / 2-\psi)$, one should find $\beta_{C}$ and $\beta_{T}$ (the pitch angles of the Chaser and Target, respectively), such that

$$
\begin{equation*}
-\rho_{C} u_{C}+\rho_{T} u_{T}=u_{\mathrm{des}} \tag{27}
\end{equation*}
$$

If the constraints (26) are not satisfied, there are no $\beta_{C} \in \mathbb{R}$ and $\beta_{T} \in \mathbb{R}$ that satisfy Eq. (27). This case will be addressed in the sequel.

## Effects of the Uncertainties in the Implementation

In order to find $\beta_{C}$ and $\beta_{T}$ that solve Eq. (27), a model for the atmospheric density must be selected*, from which the densities for the Chaser and Target are respectively assumed to be $\rho_{C_{\text {model }}}$, $\rho_{T_{\text {model }}}$. Similarly, one should also consider a nominal value for the drag coefficient $C_{D}^{*}$, which leads to a nominal value $C_{B 0}^{*}=S_{0} C_{D}^{*} /(2 m)$, yielding assumed $u_{C}^{*}=C_{B 0}^{*} \cos \left(\beta_{C}-\psi\right)$ and $u_{T}^{*}=C_{B 0}^{*} \cos \left(\beta_{T}-\psi\right)$. Hence, if

$$
\begin{equation*}
\left(-\rho_{C_{\text {model }}}+\rho_{T_{\text {model }}} \zeta\right) C_{B 0}^{*} \leq u_{\text {des }} \leq\left(-\rho_{C_{\text {model }}} \zeta+\rho_{T_{\text {model }}}\right) C_{B 0}^{*} \tag{28}
\end{equation*}
$$

$\beta_{C}$ and $\beta_{T}$ are actually found from the equation

$$
\begin{equation*}
C_{B 0}^{*}\left(-\rho_{C_{\text {model }}} \cos \left(\beta_{C}-\psi\right)+\rho_{T_{\text {model }}} \cos \left(\beta_{T}-\psi\right)\right)=u_{\text {des }} \tag{29}
\end{equation*}
$$

However, considering that the models are uncertain, the real densities affecting the spacecraft are given by $\rho_{C}=\rho_{C_{\text {model }}}+\delta_{\rho C}$ and $\rho_{T}=\rho_{T_{\text {model }}}+\delta_{\rho T}$, where $\delta_{\rho C}$ and $\delta_{\rho T}$ denote the differences between the real and modeled density, for Chaser and Target respectively. Correspondingly, the real value of $C_{B 0}$ is given by $C_{B 0}=C_{B 0}^{*}+\delta C_{B 0}$, where $\delta C_{B 0}=S_{0} \delta C_{D} /(2 m)$. This leads to $u_{C}=u_{C}^{*}+\delta u_{C}$ and $u_{T}=u_{T}^{*}+\delta u_{T}$, where $\delta u_{C / T}=\delta C_{B 0} \cos \left(\beta_{C / T}-\psi\right)$.

Hence, the true input of the system is given by

$$
\begin{align*}
u_{\text {true }} & =-\left(\rho_{C_{\text {model }}}+\delta_{\rho C}\right)\left(u_{C}^{*}+\delta u_{C}\right)+\left(\rho_{T_{\text {model }}}+\delta_{\rho T}\right)\left(u_{T}^{*}+\delta u_{T}\right) \\
& =u_{\text {des }}+\eta \tag{30}
\end{align*}
$$

where

$$
\begin{equation*}
\eta \triangleq-\rho_{C_{\text {model }}} \delta u_{C}+\rho_{T_{\text {model }}} \delta u_{T}-\delta_{\rho C} u_{C}^{*}+\delta_{\rho T} u_{T}^{*}-\delta_{\rho C} \delta u_{C}+\delta_{\rho T} \delta u_{T} \tag{31}
\end{equation*}
$$

To assess the effects on the maneuvers caused by the errors in the density models and $C_{B 0}$, consider the positive definite function $V=\mathbf{w}^{\top} \mathbf{P} \mathbf{w}^{\dagger}$. Its time derivative is then given by

$$
\begin{equation*}
\dot{V}=2 \mathbf{w}^{\top} \mathbf{P}\left[\mathbf{A} \mathbf{w}+\mathbf{B}\left(-\left(\rho_{C_{\text {model }}}+\delta_{\rho C}\right)\left(u_{C}^{*}+\delta u_{C}\right)+\left(\rho_{T_{\text {model }}}+\delta_{\rho T}\right)\left(u_{T}^{*}+\delta u_{T}\right)\right)\right] \tag{32}
\end{equation*}
$$

[^2]Introducing Eqs. (25), (29) and (31), yields

$$
\begin{align*}
\dot{V} & =2 \mathbf{w}^{\top} \mathbf{P}\left[\mathbf{A w}+\mathbf{B}\left(u_{\mathrm{des}}+\eta\right)\right]=\mathbf{w}^{\top}\left[\mathbf{P}(\mathbf{A}-\mathbf{B} \mathbf{K})+(\mathbf{A}-\mathbf{B} \mathbf{K})^{\top} \mathbf{P}\right] \mathbf{w}+2 \mathbf{w}^{\top} \mathbf{P B} \eta \\
& =-\mathbf{w}^{\top}\left(\mathbf{Q}+\mathbf{P B} R^{-1} \mathbf{B}^{\top} \mathbf{P}\right) \mathbf{w}+2 \mathbf{w}^{\top} \mathbf{P B} \eta \tag{33}
\end{align*}
$$

which can be upper bounded by

$$
\begin{equation*}
\dot{V} \leq \Upsilon(\|\mathbf{w}\|) \triangleq-\|\mathbf{w}\|^{2} \lambda_{\min }+2\|\mathbf{w}\|\|\mathbf{P B}\| \bar{\eta} \tag{34}
\end{equation*}
$$

where

$$
\begin{equation*}
\bar{\eta} \triangleq \delta C_{B 0}^{M} \rho_{M_{\text {model }}}+2 C_{B 0}^{*} \delta_{\rho}^{M}+2 \delta C_{B 0}^{M} \delta_{\rho}^{M} \geq|\eta| \tag{35}
\end{equation*}
$$

with $\delta C_{B 0}^{M} \geq\left|\delta C_{B 0}\right|, \delta_{\rho}^{M} \geq\left|\delta_{\rho_{C / T}}\right|, \rho_{\text {model }}^{M} \geq \rho_{C / T_{\text {model }}}$ along the entire trajectory. Moreover, $\lambda_{\text {min }}>0$ denotes the smallest eigenvalue of the symmetric matrix $\boldsymbol{\Xi} \triangleq \mathbf{Q}+\mathbf{P B} R^{-1} \mathbf{B}^{\top} \mathbf{P}$.

Considering the errors in the atmospheric density models and in the ballistic coefficients, $\dot{V}$ will be always bounded from above by $\Upsilon(\|\mathbf{w}\|)$, which is a parabola in $\|\mathbf{w}\|$. Its roots are located at $\|\mathbf{w}\|=0$ and $\|\mathbf{w}\|=2 \frac{\|\mathbf{P} \mathbf{B}\|}{\lambda_{\min }} \bar{\eta}>0$. Hence, as long as $\|\mathbf{w}\|>2 \frac{\|\mathbf{P} \mathbf{B}\|}{\lambda_{\min }} \bar{\eta}$, $\dot{V}$ will be negative, as required for convergence [23, Chapter 4]. Since $V$ is positive definite, due to continuity and as long as $\dot{V}<0$, eventually $\|\mathbf{w}\|$ becomes $2 \frac{\|\mathbf{P} \mathbf{B}\|}{\lambda_{\min }} \bar{\eta}$ and the decreasing rate of $V$ cannot be guaranteed. Hence, there is interest in reducing the ratio $\|\mathbf{P} \mathbf{B}\| / \lambda_{\min }$, which is a function of $\mathbf{Q}$ and $R$, to reduce the range in which $\dot{V}<0$ cannot be guaranteed.

The matrix $\boldsymbol{\Xi}=\mathbf{Q}+\mathbf{P B} R^{-1} \mathbf{B}^{\top} \mathbf{P}$ is given by

$$
\boldsymbol{\Xi}=\left[\begin{array}{ll}
\Xi_{1} & \Xi_{2}  \tag{36}\\
\Xi_{2} & \Xi_{3}
\end{array}\right]
$$

where

$$
\begin{equation*}
\Xi_{1}=2 q_{1} \quad \Xi_{2}=-\sqrt{\frac{q_{1}}{b}} \sqrt{2 P_{0} \sqrt{q_{1} R}+q_{2} b} \quad \Xi_{3}=2 q 2+\frac{2 P_{0} \sqrt{q_{1} R}}{b} \tag{37}
\end{equation*}
$$

from which its eigenvalues are computed as:

$$
\begin{equation*}
\lambda_{\max , \min }=\frac{q_{2} b+P_{0} \sqrt{q_{1} R}+q_{1} b \pm \sqrt{q_{2}^{2} b^{2}+2 q_{2} b P_{0} \sqrt{q_{1} R}-q_{1} q_{2} b^{2}+q_{1} R P_{0}^{2}+q_{1}^{2} b^{2}}}{b} \tag{38}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\frac{\|\mathbf{P} \mathbf{B}\|}{\lambda_{\min }}=\frac{\sqrt{\tilde{q}_{1}+\tilde{q}_{2}+2 \frac{P_{0}}{b} \sqrt{\tilde{q}_{1}}}}{\left(\tilde{q}_{1}+\tilde{q}_{2}\right)+\frac{P_{0}}{b} \sqrt{\tilde{q}_{1}}-\sqrt{\left(\tilde{q}_{1}^{2}+\tilde{q}_{2}^{2}\right)+\left(\frac{P_{0}}{b}\right)^{2} \tilde{q}_{1}+2 \frac{P_{0}}{b} \sqrt{\tilde{q}_{1}} \tilde{q}_{2}-\tilde{q}_{1} \tilde{q}_{2}}} \tag{39}
\end{equation*}
$$

where $\tilde{q}_{1} \triangleq q_{1} / R$ and $\tilde{q}_{2} \triangleq q_{2} / R$. Notice that the RHS of Eq. (39) does not depend on $q_{1}, q_{2}$ or $R$ explicitly, but on $\tilde{q}_{1}$ and $\tilde{q}_{2}$. Moreover, $\|\mathbf{P} \mathbf{B}\| / \lambda_{\min }$ is a positive function of $\tilde{q}_{1}$ and $\tilde{q}_{2}$, which can be selected to reduce $\|\mathbf{P} \mathbf{B}\| / \lambda_{\text {min }}$ as much as desired, while keeping in mind that this affects the required control efforts and might generate saturation in the system. Yet, the next section shows that if the system is saturated, it will eventually reach a non-saturated configuration where Eq. (28) is satisfied, and thus convergence up to $\|\mathbf{w}\|=2 \frac{\|\mathbf{P B}\|}{\lambda_{\text {min }}} \bar{\eta}>0$ can be achieved.

## Saturation

According to the proposed controller, the desired input of the system is given by Eq. (25). On the other hand, the required attitude angles $\beta_{C}$ and $\beta_{T}$ are computed according to Eq. (29). As previously mentioned, in order to find $\beta_{C} \in \mathbb{R}$ and $\beta_{T} \in \mathbb{R}$, the inequality (28) must hold. However, for certain values of $\Delta \bar{\theta}$ and $\Delta \bar{a}$ the inequality (28) might be not satisfied, and thus there are no valid $\beta_{C}$ and $\beta_{T}$ satisfying Eqs. (25) and (29). This occurs because the cross-sectional areas of the satellites are limited, whereas the input of a linear controller of the form of Eq. (25) can, theoretically, attain any value.

To address this problem, this work proposes to implement a saturation function. Whenever the inequality (28) is not satisfied, the system implements attitudes $\beta_{C}$ and $\beta_{T}$ such that the maximum or minimum feasible differential drag acceleration is obtained.

$$
\begin{array}{ll}
\beta_{C}=\frac{\pi}{2} \text { and } \beta_{T}=\psi, & \text { if }\left(-k_{1} \Delta \bar{\theta}-k_{2} \bar{a}\right) \geq C_{B 0}\left(-\rho_{C_{\text {model }}} \zeta+\rho_{T_{\text {model }}}\right) \\
\beta_{C}=\psi \text { and } \beta_{T}=\frac{\pi}{2}, & \text { if }\left(-k_{1} \Delta \bar{\theta}-k_{2} \bar{a}\right) \leq C_{B 0}\left(-\rho_{C_{\text {model }}}+\rho_{\left.T_{\text {model }} \zeta\right)} \zeta\right) \\
C_{B 0}^{*}\left(-\rho_{C_{\text {model }}} \cos \left(\beta_{C}-\psi\right)+\rho_{T_{\text {model }}} \cos \left(\beta_{T}-\psi\right)\right)=-\mathbf{K}[\Delta \bar{\theta}, \Delta \bar{a}]^{\top}, \text { otherwise } \tag{42}
\end{array}
$$

where $k_{1}$ and $k_{2}$ respectively denote the first and second components of the gain matrix $\mathbf{K}$, given by Eq. (24).

The goal of this section is showing that the dynamical system given by Eqs. (14) and (17) still converges to the origin, if it is driven by the control law stated by Eqs. (40-42). This will be done by depicting the phase portrait of the system and analysing the resulting trajectories. Since the atmospheric density certainly depends on the altitude of the satellites, it is necessary to assume a density field model that captures the main density behavior due to changes in the semi-major axes of the satellites. For the forthcoming analysis, the exponential atmospheric model CIRA-72 published in [1, Page 564] will be used, as it captures the aforementioned behavior and allows to keep the math tractable. In this model, the atmospheric density is computed as

$$
\begin{equation*}
\rho(h)=\rho_{H} e^{-\frac{h-h_{0}}{H}} \tag{43}
\end{equation*}
$$

where $\rho_{H}, h_{0}$, and $H$ denote model parameters that are tabulated in the aforementioned reference, for various intervals of altitude $h$. Notice that this model, and other similar static models, do not consider variations of the density due to the local hour of the satellite. They are built averaging density measurements for given altitudes. As DD maneuvers require large numbers of orbits to be completed, the oscillations due to local time tend to average out. Hence, for the purposes of this section, the main effects of the maneuvers can still be captured utilizing the aforementioned model.

Since the orbits are assumed circular (assuming circular Earth with radius $R_{e q}$ ), Eq. (43) can be restated as

$$
\begin{equation*}
\rho(\bar{a})=\rho_{H} e^{-\frac{\bar{a}-\left(h_{0}+R_{e q}\right)}{H}} \tag{44}
\end{equation*}
$$

where the constant $h_{0}$ represents the lowest altitude of the interval of interest. Since the mean semi-major axes of the satellites are expected to be sufficiently close, from Eq. (44), $\rho_{C_{\text {model }}}$ can be modeled as

$$
\begin{equation*}
\rho_{C_{\text {model }}} \simeq \rho_{T_{\text {model }}}+\left.\frac{\partial \rho}{\partial \bar{a}}\right|_{\bar{a}_{T}} \Delta \bar{a}=\rho_{T_{\text {model }}}\left(1-\frac{\Delta \bar{a}}{H}\right) \tag{45}
\end{equation*}
$$

where $H$ is selected for the proper range of altitudes.*
To proceed, the region in which Eq. (42) has solutions $\beta_{C} \in \mathbb{R}$ and $\beta_{T} \in \mathbb{R}$ is firstly determined. From the inequality (28) and introducing Eq. (45), this region is obtained as

$$
\begin{equation*}
S_{U} \Delta \bar{\theta}+M_{U} \geq \Delta \bar{a} \geq M_{L}+S_{L} \Delta \bar{\theta} \tag{46}
\end{equation*}
$$

where
$S_{U} \triangleq \frac{-k_{1}}{k_{2}+\frac{C_{B 0}}{H} \rho_{T_{\text {model }}}}>0 \quad S_{L} \triangleq \frac{-k_{1}}{k_{2}+\frac{C_{B 0}}{H} \zeta \rho_{T_{\text {model }}}}>0 \quad M_{U} \triangleq \frac{(1-\zeta) C_{B 0}}{\frac{k_{2}}{\rho_{T_{\text {model }}}}+\frac{C_{B 0}}{H}}>0 \quad M_{L} \triangleq-\frac{C_{B 0}(1-\zeta)}{\frac{k_{2}}{\rho_{T_{\text {model }}}}+\frac{C_{B 0}}{H} \zeta}<0$
(47)

Notice that, for a given $\rho_{T_{\text {model }}}$, $\Delta \bar{a}$ is bounded by an upper line $\mathcal{L}_{U}$ and a lower line $\mathcal{L}_{L}$, of positive slopes. Figure 2 shows these lines (dash-dot) for typical values of the parameters of the inequality (46), namely: $C_{D}=2.2, C_{B 0}=0.0134 \mathrm{~m}^{2} / \mathrm{kg}, S_{A}=0.06 \mathrm{~m}^{2}$ and $S_{B}=0.01 \mathrm{~m}^{2}, m=5$ $\mathrm{kg}, k_{1}=-1.830310^{-10} 1 / \mathrm{km}$ and $k_{2}=1.853610^{-10} 1 / \mathrm{km}^{2}, H=58.5150 \mathrm{~km},{ }^{1}$ and $\rho_{T_{\text {model }}}=$ $2.563410^{-12} \mathrm{~kg} / \mathrm{m}^{3}$, corresponding to an altitude of $421.87 \mathrm{~km} .{ }^{1}$ For the following explanation, the zone in between $\mathcal{L}_{U}$ and $\mathcal{L}_{L}$ will be referred to as non-saturated zone.

Moreover, Fig. 2 also illustrates the solid line $\mathcal{O}$ that satisfies $\Delta \dot{\bar{a}}=0$. This line is obtained by solving $k_{1} \Delta \bar{\theta}+k_{2} \Delta \bar{a}=0$. The slope of this line is $-k_{1} / k_{2}>0$, and it passes through the origin. Above(below) $\mathcal{O}, \Delta \dot{\bar{a}}<(>) 0$. Furthermore, notice that above (below) the line $\Delta \bar{a}=0$, $\Delta \dot{\bar{\theta}}<(>) 0$.

According to Eq. (17), there are two necessary conditions that must hold in order to be able to generate differential drag for control purposes. These conditions are formulated as $-\rho_{C_{\text {model }}}+$ $\rho_{T_{\text {model }}} \zeta<0$ and $-\rho_{C_{\text {model }}} \zeta+\rho_{T_{\text {model }}}>0$. If one of these conditions does not hold, $\Delta \dot{\bar{a}}$ could not be generated in both directions, positive and negative. Considering Eq. (45), these conditions entail the following requirements: $1>\zeta+\Delta \bar{a} / H$ and $1>\zeta(1-\Delta \bar{a} / H)$. As it was previously mentioned, the expected values of $\Delta \bar{a}$ are sufficiently small to avoid high natural drift rates, from which for the purposes of this work they can be assumed as $|\Delta \bar{a}| \leq 10 \mathrm{~km}$. Moreover, for 3 U or longer cubesats, $\zeta=\cos (\pi / 2-\psi) \leq 0.32$. Hence, $1>\zeta+\Delta \bar{a} / H$ and $1>\zeta(1-\Delta \bar{a} / H)$ do not appear difficult to satisfy. If these conditions were not satisfied, the non-saturated zone would not exist, nor the line $\mathcal{O}$ could be defined.

Figure 2 also depicts the phase portrait of the system (18) driven by (40-42). At each point $\mathbf{w}=[\Delta \bar{\theta}, \Delta \bar{a}]^{\top}$, the slope of the flow direction is computed as $\frac{\mathrm{d} \Delta \bar{a}}{\mathrm{~d} \Delta \theta}=\Delta \dot{\bar{a}} / \Delta \dot{\bar{\theta}}$. In the nonsaturated zone, the slope of the flow direction is given by

$$
\begin{equation*}
\frac{\mathrm{d} \Delta \bar{a}}{\mathrm{~d} \Delta \bar{\theta}}=\frac{2 \sqrt{\mu a_{T}}}{P_{0}}\left(k_{1} \frac{\Delta \bar{\theta}}{\Delta \bar{a}}+k_{2}\right) \tag{48}
\end{equation*}
$$

whereas out of this region, the slopes are given by

$$
\begin{align*}
\frac{\mathrm{d} \Delta \bar{a}}{\mathrm{~d} \Delta \bar{\theta}} & =-\frac{2 \sqrt{\mu a_{T}}}{P_{0} \Delta \bar{a}}\left(C _ { B 0 } \left(-\rho_{C_{\text {model }}}+\rho_{\left.\left.T_{\text {model }} \zeta\right)\right)}\right.\right. \\
& =-\frac{2 \sqrt{\mu a_{T}} C_{B 0} \rho_{T_{\text {model }}}\left(\zeta-\left(1-\frac{\Delta \bar{a}}{H}\right)\right)}{P_{0} \Delta \bar{a}}, \quad \text { if } \mathbf{w} \text { is above } \mathcal{L}^{U} \tag{49}
\end{align*}
$$

[^3]and
\[

$$
\begin{align*}
\frac{\mathrm{d} \Delta \bar{a}}{\mathrm{~d} \Delta \bar{\theta}} & =-\frac{2 \sqrt{\mu a_{T}}}{P_{0} \Delta \bar{a}}\left(C_{B 0}\left(-\rho_{C_{\text {model }}} \zeta+\rho_{T_{\text {model }}}\right)\right) \\
& =-\frac{2 \sqrt{\mu a_{T}} C_{B 0} \rho_{T_{\text {model }}}\left(-\zeta\left(1-\frac{\Delta \bar{a}}{H}\right)+1\right)}{P_{0} \Delta \bar{a}}, \quad \text { if } \mathbf{w} \text { is below } \mathcal{L}^{L} \tag{50}
\end{align*}
$$
\]



Figure 2. Phase Portrait of the System, depicted with Eqs. (48-50).
The line defined as $\Delta \bar{a}=0$ along with the line $\mathcal{O}$ determine four regions of the phase portrait, each of which has distinctive flow direction. These regions are named I, II, III, and IV. In region I, the flow has $\Delta \dot{\bar{a}}>0$ and $\Delta \dot{\bar{\theta}}<0$. In region II, the flow evolves such that $\Delta \dot{\bar{a}}<0$ and $\Delta \dot{\bar{\theta}}<0$. In region III, the flow is characterized by $\Delta \dot{\bar{a}}<0$ and $\Delta \dot{\bar{\theta}}>0$. Finally, in region IV, the flow moves with $\Delta \dot{\bar{a}}>0$ and $\Delta \dot{\bar{\theta}}>0$. Each of these regions has a sub-region within the non-saturated zone, denoted by $(\cdot)^{\mathcal{N}}$, and a sub-region that is outside the non-saturated zone, which is denoted by $(\cdot)^{\mathcal{S}}$. Hence, for instance, region I outside(inside) the non-saturated zone will be referred to as $I^{\mathcal{S}(\mathcal{N})}$.

As time elapses, considering the slow decay of $\bar{a}_{T}$ and consequent variations of $\rho_{T_{\text {model }}}$ and $P_{0}$, the slopes outside the non-saturated zone become steeper. To see that, using Eqs. (15,44,49,50), one can compute that:

$$
\operatorname{sgn}\left[\frac{\partial}{\partial \bar{a}_{T}}\left(\frac{\mathrm{~d} \Delta \bar{a}}{\mathrm{~d} \Delta \bar{\theta}}\right)\right]= \begin{cases}-1 \operatorname{sgn}(\Delta \bar{a}), & \text { if } \mathbf{w} \text { is above } \mathcal{L}^{U}  \tag{51}\\ +1 \operatorname{sgn}(\Delta \bar{a}), & \text { if } \mathbf{w} \text { is below } \mathcal{L}^{L}\end{cases}
$$

Hence, since $\dot{\bar{a}}_{T}<0$, the slopes increase their magnitudes.
Due to the flow directions of regions $\mathcal{I}^{\mathcal{S}}$ and $\mathrm{III}^{\mathcal{S}}$, the trajectories will eventually reach the nonsaturated zone. In region $\mathrm{IV}^{\mathcal{S}}$, as long as

$$
\begin{equation*}
\frac{\mathrm{d} \Delta \bar{a}}{\mathrm{~d} \Delta \bar{\theta}}>S_{L} \tag{52}
\end{equation*}
$$

the trajectories will reach either the region $\mathrm{IV}^{\mathcal{N}}$ or the region $\mathrm{I}^{\mathcal{S}}$, in which case they also end up at the non-saturated zone. Inequality (52), entails

$$
\begin{equation*}
\Delta \bar{a}>\frac{\zeta-1}{\frac{P_{0} S_{L}}{2 \sqrt{\mu a_{T}} C_{B 0} \rho_{T_{\text {model }}}}+\frac{\zeta}{H}} \tag{53}
\end{equation*}
$$

and with the same typical values that Fig. 2 was built, inequality (52) is satisfied as long as $\Delta \bar{a}>$ -22.03 km .

In region $\mathrm{II}^{\mathcal{S}}$, as long as

$$
\begin{equation*}
\frac{\mathrm{d} \Delta \bar{a}}{\mathrm{~d} \Delta \bar{\theta}}>S_{U} \tag{54}
\end{equation*}
$$

the trajectories will flow towards the region $\mathrm{II}^{\mathcal{N}}$ or $\mathrm{III}^{\mathcal{S}}$ in which case they will eventually reach the non-saturated zone as well. Condition (54) implies

$$
\begin{equation*}
\Delta \bar{a}<\frac{1-\zeta}{\frac{P_{0} S_{U}}{2 \sqrt{\mu_{T} C_{B}} C_{B 0} \rho_{\text {model }}}+\frac{1}{H}} \tag{55}
\end{equation*}
$$

With the same values used to build Fig. 2, inequality (54) is satisfied as long as $\Delta \bar{a}<16.06 \mathrm{~km}$. For practical purposes and due to aforesaid reasons, the value of $|\Delta \bar{a}|$ is expected to be significantly smaller than $\sim 16 \mathrm{~km}$.

Since the phase portrait shows that the trajectories always move towards and eventually enter the non-saturated zone, they cannot leave once they reached it. Therefore, once the trajectories are within the non-saturated zone, the LQR controller drives the system towards the origin, with no saturation. A sample trajectory, generated with the parameters used to build Fig. 2, is also shown (thicker solid lines) to illustrates the aforementioned concepts.

## Computation of $\boldsymbol{\beta}_{\mathrm{C}}$ and $\boldsymbol{\beta}_{\mathbf{T}}$

Using Eqs. (40-42), the angles $\beta_{C}$ and $\beta_{T}$ are determined. If the system is within the nonsaturated zone, then $\beta_{C}$ and $\beta_{T}$ must satisfy Eq. (42), which constitutes a single equation with two unknowns. In order to minimize the orbital decay, it is sought that the cross-sectional areas are always as small as possible. Therefore, one of the angles may be arbitrarily set to yield the minimum possible cross-sectional area, and the other one determined to solve Eq. (42). Hence, the following algorithm is proposed to determine $\beta_{C}$ and $\beta_{T}$.

$$
\left\{\begin{array}{l}
\beta_{C}=90^{\circ} \text { and } \beta_{T}=90^{\circ},  \tag{56}\\
\beta_{C}=90^{\circ} \text { and } \beta_{T}=\arccos \left[\frac{-\mathbf{K}[\Delta \bar{\theta}, \bar{a}]^{\top}+\rho_{C_{\text {model }}} C_{B 0} \zeta}{C_{B 0} \rho_{T_{\text {model }}}}\right]+\psi, \quad \text { if } \bar{a}_{C}=\bar{a}_{T} \text { and } \bar{\theta}_{C}=\bar{\theta}_{T} \\
\beta_{T}=90^{\circ} \text { and } \beta_{C}=\arccos \left[\frac{\mathbf{K w} \geq 0}{\mathbf{K}_{B 0}[\Delta \bar{\theta}, \bar{a}]^{\top}+\rho_{T_{\text {model }}} C_{B 0} \zeta}\right]+\psi, \quad \text { otherwise }
\end{array}\right.
$$

## MIN-MAX CONTROL FOR SHORT RANGE DIFFERENTIAL DRAG BASED MANEUVERS WITH INPUT UNCERTAINTIES

In this section a second methodology for DD coplanar relative maneuvering is presented. The methodology is intended for close proximity initial conditions, for instance once the system has reached the bound given in Eq. (34). In this approach the relative motion is described using cartesian coordinates in a Local Vertical-Local Horizontal (LVLH) frame, attached to the Target. In this frame, $x$ points from the Earth's center to the Target spacecraft, $y$ points in the direction of the Target's velocity vector and $z$ completes the right handed frame. The use of Cartesian coordinates enables a more precise description of the relative state for short distances. Moreover, estimation of the relative state is more accurate, since differential GPS can provide direct measurements of the relative state in the Cartesian frame [24, Page 7]. The relative motion of the spacecraft is controlled
using a min-max controller ${ }^{16}$ which takes into account uncertainties in the input, provided that they are bounded by a known value. In this case the controller provides asymptotic stability under bounded uncertainties in the drag force.

## Spacecraft Linearized Relative Dynamics

The relative motion is represented using the dynamical model developed by Schweighart and Sedwick. ${ }^{10}$ This relative motion model considers that the satellites are affected by the mean effects of $J_{2}$, and that the Target satellite is in a circular orbit or that there is a circular reference orbit. For motion restricted to the $x y$ plane, this model is written as:

$$
\dot{\boldsymbol{v}}=\mathbf{A}_{m} \boldsymbol{v}+\mathbf{B}_{m} w_{\text {true }}, \quad \mathbf{A}_{m}=\left[\begin{array}{cccc}
0 & 0 & 1 & 0  \tag{57}\\
0 & 0 & 0 & 1 \\
b & 0 & 0 & a \\
0 & 0 & -a & 0
\end{array}\right], \quad \boldsymbol{v}=\left[\begin{array}{l}
x \\
y \\
\dot{x} \\
\dot{y}
\end{array}\right]
$$

where the product $\mathbf{B}_{m} w_{\text {true }}$ denotes the relative acceleration of the Chaser with respect to the Target, resolved in the LVLH frame, and $a$ and $b$ are given by:

$$
\begin{equation*}
a=2 n c, \quad b=\left(5 c^{2}-2\right) n^{2}, \quad c=\sqrt{1+\frac{3 J_{2} R_{e q}^{2}}{8 \bar{a}_{r e f}^{2}}\left(1+3 \cos \left(2 \bar{i}_{r e f}\right)\right)} \tag{58}
\end{equation*}
$$

with $n$, $\bar{a}_{\text {ref }}$ and $\bar{i}_{\text {ref }}$ being the mean mean motion, mean semi-major axis and mean inclination of a circular reference orbit, respectively. In this work the reference orbit is set to match the orbit of the Target. During the maneuver, the drag acceleration results in orbital decay for the Target and Chaser. Yet, the variations in the semi-major axes due to the decay are significantly small, which justify considering the matrix $\mathbf{A}_{m}$ as a constant and determined by the initial conditions*. Numerical simulations will illustrate the small changes in the entries of $\mathbf{A}_{m}$.

If the satellites are close enough, such that the inertial velocity vectors of Chaser and Target are practically parallel, the differential acceleration caused by drag is given by $\mathbf{B}_{m} w_{\text {true }}$ with $\mathbf{B}_{m}=$ [0001] ${ }^{\top}$, and

$$
\begin{equation*}
w_{\text {true }}=-\rho_{C}\left\|\mathbf{v}_{C}\right\|^{2} u_{C}+\rho_{T}\left\|\mathbf{v}_{T}\right\|^{2} u_{T} \tag{59}
\end{equation*}
$$

Considering uncertainties in the atmospheric density and drag coefficient (as it was done in the previous section in Eq. (30)), Eq. (59) is reformulated as

$$
\begin{align*}
w_{\text {true }} & =-\left(\rho_{C_{\text {model }}}+\delta_{\rho_{C}}\right)\left\|\mathbf{v}_{C}\right\|^{2}\left(u_{C}^{*}+\delta u_{C}\right)+ \\
& +\left(\rho_{T_{\text {model }}}+\delta_{\rho_{T}}\right)\left\|\mathbf{v}_{T}\right\|^{2}\left(u_{T}^{*}+\delta u_{T}\right)  \tag{60}\\
& =w^{*}+\eta_{w} \tag{61}
\end{align*}
$$

where $w^{*}$ contains the known part of the differential drag acceleration, given by

$$
\begin{equation*}
w^{*}=-\rho_{C_{\text {model }}}\left\|\mathbf{v}_{C}\right\|^{2} u_{C}^{*}\left(\beta_{C}\right)+\rho_{T_{\text {model }}}\left\|\mathbf{v}_{T}\right\|^{2} u_{T}^{*}\left(\beta_{T}\right) \tag{62}
\end{equation*}
$$

and $\eta_{w}$ includes the uncertain terms

$$
\begin{align*}
\eta_{w}= & -\left\|\mathbf{v}_{C}\right\|^{2}\left(\rho_{C_{\text {model }}} \delta u_{C}+\delta_{\rho C} u_{C}^{*}+\delta u_{C} \delta_{\rho C}\right)+ \\
& +\left\|\mathbf{v}_{T}\right\|^{2}\left(\rho_{T_{\text {model }}} \delta u_{T}+\delta_{\rho T} u_{T}^{*}+\delta u_{T} \delta_{\rho T}\right) \tag{63}
\end{align*}
$$

*This practice was also adopted in former publications. ${ }^{5,9,11}$

The dynamical model is then stated as:

$$
\begin{equation*}
\dot{\boldsymbol{v}}=\mathbf{A}_{m} \boldsymbol{v}+\mathbf{B}_{m}\left(w^{*}+\eta_{w}\right) \tag{64}
\end{equation*}
$$

If the uncertainties in the density and the drag coefficient are bounded, then there is a bound for $\eta_{w}$ such that $\left\|\eta_{w}\right\| \leq \bar{\eta}_{w}$. To find such a bound, Eq. (63) is rewritten as

$$
\begin{equation*}
\eta_{w}=\eta_{w 1}+\eta_{w 2}+\eta_{w 3} \tag{65}
\end{equation*}
$$

where

$$
\begin{gather*}
\eta_{w 1} \triangleq-\left\|\mathbf{v}_{C}\right\|^{2} \rho_{C_{\text {model }}} \delta u_{C}+\left\|\mathbf{v}_{T}\right\|^{2} \rho_{T_{\text {model }}} \delta u_{T}  \tag{66}\\
\eta_{w 2} \triangleq-\left\|\mathbf{v}_{C}\right\|^{2} \delta_{\rho C} u_{C}^{*}+\left\|\mathbf{v}_{T}\right\|^{2} \delta_{\rho T} u_{T}^{*} \tag{67}
\end{gather*}
$$

and

$$
\begin{equation*}
\eta_{w 3} \triangleq-\left\|\mathbf{v}_{C}\right\|^{2} \delta u_{C} \delta_{\rho C}+\left\|\mathbf{v}_{T}\right\|^{2} \delta u_{T} \delta_{\rho T} \tag{68}
\end{equation*}
$$

Defining $\xi_{C / T} \triangleq\left\|\mathbf{v}_{C / T}\right\|^{2} \rho_{C / T_{\text {model }}}, \xi_{\max } \triangleq \max \left[\xi_{C}, \xi_{T}\right]$, and $\xi_{\min } \triangleq \min \left[\xi_{C}, \xi_{T}\right], \eta_{w 1}$ can be bounded as

$$
\begin{equation*}
\left|\eta_{w 1}\right| \leq\left(\xi_{\max }-\xi_{\min } \zeta\right) \delta C_{B 0}^{M} \tag{69}
\end{equation*}
$$

where $\delta C_{B 0}^{M} \geq\left|\delta C_{B 0}\right|$. To find a bound on $\eta_{w 2}$, one can consider that as the satellites are expected to be sufficiently close, the density modeling errors are expected to have the same signs. For instance, if there is an increment in the density due to a diurnal variation (both satellite in the illuminated side of Earth), then the sign of the error will be positive for both; the same can be considered if the errors are due to solar storms. Hence, it will be assumed that $\operatorname{sign}\left(\delta_{\rho C}\right)=\operatorname{sign}\left(\delta_{\rho T}\right)$. Hence,

$$
\begin{equation*}
\left|\eta_{w 2}\right| \leq C_{B 0}^{*}\left|\delta_{\rho}^{M}\right|\left\|\mathbf{v}_{\max }\right\|^{2} \tag{70}
\end{equation*}
$$

where $\left\|\mathbf{v}_{\max }\right\| \triangleq \max \left[\left\|\mathbf{v}_{C}\right\|,\left\|\mathbf{v}_{T}\right\|\right]$ and $\delta_{\rho}^{M} \geq\left|\delta_{\rho_{C / T}}\right|$. Using the same aforesaid argument

$$
\begin{equation*}
\left|\eta_{w 3}\right| \leq \delta C_{B 0}^{M}\left\|\mathbf{v}_{\max }\right\|^{2} \delta_{\rho}^{M} \tag{71}
\end{equation*}
$$

Finally, since $\eta_{w} \leq\left|\eta_{w 1}\right|+\left|\eta_{w 2}\right|+\left|\eta_{w 3}\right|, \bar{\eta}_{w}$ is set to be:

$$
\begin{equation*}
\bar{\eta}_{w}=\left(\xi_{\max }-\xi_{\min } \zeta\right) \delta C_{B 0}^{M}+C_{B 0}^{*}\left|\delta_{\rho}^{M}\right|\left\|\mathbf{v}_{\max }\right\|^{2}+\delta C_{B 0}^{M}\left\|\mathbf{v}_{\max }\right\|^{2} \delta_{\rho}^{M} \tag{72}
\end{equation*}
$$

## Min-Max Controller

Gutman ${ }^{16}$ presented a min-max control law that provides asymptotic stability for a linear system with uncertainties in the input. This control law can be formulated as:

$$
\begin{equation*}
u_{M M}=-\mathbf{K}_{P} \boldsymbol{v}-\bar{u} \frac{\alpha}{\|\alpha\|}, \quad \alpha=\mathbf{B}_{m}^{\top} \mathbf{P}_{L} \boldsymbol{v}, \quad \bar{\eta} \leq \bar{u} \tag{73}
\end{equation*}
$$

where $\mathbf{K}_{P}$ is a gain that stabilizes the corresponding linear system (such as the one in Eq. (64)) without uncertainties in the input, and matrix $\mathbf{P}_{L}$ is the solution of the following Lyapunov equation

$$
\begin{equation*}
\mathbf{P}_{L} \overline{\mathbf{A}}+\overline{\mathbf{A}}^{\top} \mathbf{P}_{L}=-\mathbf{Q}_{L}, \quad \overline{\mathbf{A}}=\mathbf{A}_{m}-\mathbf{B}_{m} \mathbf{K}_{P} \tag{74}
\end{equation*}
$$

where $\mathbf{Q}_{L}$ is a positive definite matrix.

In this work, at the initial state, $\mathbf{K}_{P}$ is determined such that $\overline{\mathbf{A}}$ is Hurwitz using the pole placement technique.

To implement the min-max control law, the one should assume bounds on the uncertainties of the density and drag coefficient, which allow calculating $\bar{\eta}_{w}$. Then $\bar{u}$ is selected to match $\bar{\eta}_{w}$, thus satisfying the requirement in Eq. (73), and producing the following control law.

$$
\begin{equation*}
w_{d e s}=-\mathbf{K}_{P} \boldsymbol{v}-\bar{\eta}_{w} \frac{\alpha}{\|\alpha\|} \tag{75}
\end{equation*}
$$

Analogously to the previous section, to compute $\beta_{C}$ and $\beta_{T}, w^{*}$ is equated to $w_{d e s}$, from which one should solve for $\beta_{C}$ and $\beta_{T}$.

$$
\begin{equation*}
-\rho_{C_{\text {model }}}\left\|\mathbf{v}_{C}\right\|^{2} u_{C}^{*}+\rho_{T_{\text {model }}}\left\|\mathbf{v}_{T}\right\|^{2} u_{T}^{*}=-\mathbf{K}_{P} \boldsymbol{v}-\bar{\eta}_{w} \frac{\alpha}{\|\alpha\|} \tag{76}
\end{equation*}
$$

Similarly to previous section (Eq. (56)), to reduce orbital decay, the $\beta$ of one of the spacecraft is set to $90^{\circ}$ and then Equation (76) is solved for the remaining $\beta$. This can be formulated as:

$$
\begin{cases}\beta_{C}=90^{\circ} \text { and } \beta_{T}=90^{\circ}, & \text { if } w_{\text {des }}=0  \tag{77}\\ \beta_{T}=90^{\circ} \text { and } \beta_{C}=\arccos \left[\frac{C_{D} S_{B} \rho_{T_{\text {model }}}\left\|v_{T}\right\|^{2}+2 m w_{\text {des }}}{C_{D} \rho_{C_{\text {model }}}\left\|\mathbf{v}_{C}\right\|^{2} S_{0}}\right]+\psi, & \text { if } w_{\text {des }}<0 \\ \beta_{C}=90^{\circ} \text { and } \beta_{T}=\arccos \left[\frac{C_{D} S_{B} \rho_{C_{\text {model }}}\left\|v_{C}\right\|^{2}+2 m w_{\text {des }}}{C_{D} \rho_{T_{\text {model }}}\left\|v_{T}\right\|^{2} S_{0}}\right]+\psi, & \text { otherwise }\end{cases}
$$

The closed loop stable dynamics is then obtained by introducing Eq. (75) into Eq. (64), yielding

$$
\begin{equation*}
\dot{\boldsymbol{v}}=\overline{\mathbf{A}} \boldsymbol{v}+\mathbf{B}_{m}\left(\eta_{w}-\bar{\eta}_{w} \frac{\alpha}{\|\alpha\|}\right) \tag{78}
\end{equation*}
$$

To study the stability of the system in Eq. ((78)), a quadratic Lyapunov function and its time derivative are formulated as follows:

$$
\begin{align*}
& V_{m}=\boldsymbol{v}^{\top} \mathbf{P}_{L} \boldsymbol{v} \\
& \dot{V}_{m}=-\boldsymbol{v}^{\top} \mathbf{Q}_{L} \boldsymbol{v}+2 \alpha \eta_{w}-2|\alpha| \bar{\eta}_{w} \tag{79}
\end{align*}
$$

By definition $V_{m}$ is positive since $\mathbf{P}_{L}$ is positive definite. Furthermore since $\mathbf{Q}_{L}$ is positive definite and $|\alpha| \bar{\eta}_{w}>\alpha \eta_{w}$ then $\dot{V}_{m}$ is negative definite (recall $\eta_{w} \leq \bar{\eta}$ ). Consequently, the origin is a stable point.

## NUMERICAL SIMULATIONS

This section illustrates the rendezvous-oriented maneuvers performed with the control laws proposed in Eqs. (40)-(42) and (75). The maneuvers are studied using two MATLAB simulations: a long-range re-phasing maneuver and a subsequent short range rendezvous maneuver.

The following parameters were common for both simulations, while the parameters that were specifically assumed for each simulations are detailed in the corresponding subsections.

The parameters of the satellites were set as $S_{A}=0.06 \mathrm{~m}^{2}, S_{B}=0.01 \mathrm{~m}^{2}, m=5 \mathrm{~kg}$. Moreover, the real drag coefficient was defined as $C_{D}=2.39$, while the nominal one was $C_{D}^{*}=2.2 .{ }^{15,25}$

Hence, the assumed ballistic coefficient was $C_{B 0}^{*}=0.0134 \mathrm{~m}^{2} / \mathrm{kg}$, while the real one was $C_{B 0}=$ $0.0146 \mathrm{~m}^{2} / \mathrm{kg}$. The uncertainty in the ballistic coefficient was bounded by $\delta C_{B 0}^{M}=0.0013 \mathrm{~m}^{2} / \mathrm{kg}$.

The used density model $\rho_{\text {model }}$ was the International Standard Atmosphere (ISA)-76, ${ }^{1}$ and it is utilized for computing $\beta_{C}$ and $\beta_{T}$, according to (56) and (77), for the first and second simulations of the maneuver respectively. Moreover, the real density utilized for propagating the orbits was arbitrarily implemented as $\rho=\rho_{C / T_{\text {model }}}+\delta_{\rho_{C / T}}$. The quantities $\delta_{\rho_{C / T}}$ basically represent periodic oscillations of positive sign, which will be described in detail for each simulation. Modeling the real density as oscillations on top of $\rho_{C / T_{\text {model }}}$ generates a density behavior that roughly resembles oscillations due to diurnal-nocturnal density variations. Moreover, the fact that they are arbitrarily set positive produces a density field, which if it were averaged along one revolution, it would be still different from the model $\rho_{C / T_{\text {model }}}$.

## Long Range Re-Phasing Maneuver

This maneuver was performed with the control law described by Eqs. (40)-(42). For this simulation, $\delta \rho_{C / T}$ was assumed as

$$
\begin{equation*}
\delta \rho_{C / T}=0.1 \rho_{C / T_{\text {model }}}\left|\sin \left(\bar{\theta}_{C / T}\right)\right| \tag{80}
\end{equation*}
$$

The amplitude of $\delta \rho_{C / T}$ was arbitrarily chosen based on Bruinsma's work ${ }^{26}$ comparing DTM2013 estimated densities with accelerometer derived densities from the CHAMP mission over a period of nine years. Moreover, $\delta_{\rho}^{M}$ was defined as $\delta_{\rho}^{M}=5 \cdot 10^{-13} \mathrm{~kg} / \mathrm{m}^{3}$, which satisfies $\delta_{\rho}^{M}>0.1 \rho_{C / T_{\text {model }}}$ for the entire maneuver.

Recalling Eq. (35) and taking a value of $\rho_{\text {model }}^{M}$ as $\rho_{\text {model }}^{M} \triangleq 4.46 \cdot 10^{-3} \mathrm{~kg} / \mathrm{km}^{3}$, which is larger than $\rho_{C_{\text {model }}}$ and $\rho_{T_{\text {model }}}$ along the orbit, $\bar{\eta}$ is computed as $\bar{\eta}=1.98 \cdot 10^{-11} \mathrm{~km}^{-1} . \tilde{q}_{1}$ and $\tilde{q}_{2}$ are set as $\tilde{q}_{1}=\tilde{q}_{2} \triangleq 290 \cdot 10^{-19}$ resulting in $\|\mathbf{P B}\| / \lambda_{\min }=2.627 \cdot 10^{8}$, and entailing $\dot{V}<0$ as long as $\|\mathbf{w}\|>0.01$ (the quantities involved were established using km, rad, kg, and sec). Recall that this formulation is oriented to drive the satellites from significantly different positions in the orbital plane, towards a close-proximity configuration. Hence, having $\dot{V}<0$ as long as $\|\mathbf{w}\|>0.01$ is sufficient for the purpose of this maneuver.

The first simulation, performed according to Eqs. (40)-(42) is a long- range re-phasing. The initial conditions were set as $\bar{a}_{C}\left(t_{0}\right)=6800 \mathrm{~km}, \bar{a}_{T}\left(t_{0}\right)=6800.01 \mathrm{~km}, \bar{\theta}_{C}\left(t_{0}\right)=0$ and $\bar{\theta}_{T}\left(t_{0}\right)=50$ deg , and $\bar{e}_{C}\left(t_{0}\right)=\bar{e}_{T}\left(t_{0}\right)=0$. The inclination of both orbits was set as $\bar{i}_{C}=\bar{i}_{T}=1.7 \mathrm{deg}$. In order to evaluate the difference in the right ascension of the ascending nodes $\bar{\Omega}_{C}$ and $\bar{\Omega}_{T}$ caused by the Earth's oblateness $\left(J_{2}\right)$, the initial conditions for these variables were set as $\bar{\Omega}_{C}=\bar{\Omega}_{T}=0$ deg.

With the aforementioned parameters and initial conditions, a simulation was performed integrating the arguments of latitude according to Eqs. (5) and the semi-major axes according to (8) (i.e. the non-linear equations). Moreover, in order to assess the validity of the adopted dynamical model, another simulation was run utilizing cartesian elements. To that end, the following dynamical equations were integrated for each satellite

$$
\ddot{\mathbf{r}}=-\mu \frac{\mathbf{r}}{\|\mathbf{r}\|^{3}}-\frac{1}{2\|\mathbf{r}\|^{5}} \mu J_{2} R_{e q}^{2}\left[6\left[\begin{array}{l}
0  \tag{81}\\
0 \\
z
\end{array}\right]+\left(3-\frac{15 z^{2}}{\|\mathbf{r}\|^{2}}\right) \mathbf{r}\right]-C_{B 0} \rho\|\mathbf{v}\| \mathbf{v}
$$

Since the initial conditions are set in terms of mean orbital elements, the dynamic equations formulated in cartesian elements, and the controller formulated in terms of mean orbital elements,
a conversion from mean (osculating) orbital elements to osculating (mean) orbital elements should be implemented. To that end, the First Order Approximation of the Brouwer theory ${ }^{27}$ was utilized.


Figure 3. Re-phasing Maneuver: $\Delta \bar{\theta}, \Delta \bar{a}, \beta$, and $V$.


Figure 4. Re-phasing Maneuver: $\bar{a}, \Delta \Omega$, $\mathrm{u}_{\mathrm{des}}$, and Inter-satellite Distance.
Figure 3 shows the evolution of the variables $\Delta \bar{\theta}$ and $\Delta \bar{a}$, as obtained from the simulation performed integrating the mean orbital elements, and the simulation performed integrating the cartesian coordinates. Moreover, the angles $\beta_{C}$ and $\beta_{T}$ are also depicted as determined from both simula-
tions; the one in terms of mean orbital elements as well as the one in terms of cartesian coordinates. Finally, the convergence of the function $V$ is also illustrated. As expected, since $\|\mathbf{w}\|>0.01$, $\dot{V}<0$.

Figure 4 shows the decay of the mean semi-major axes, and the desired control $\mathbf{u}_{\text {des }}=-\mathbf{K w}$. Despite the long duration of the maneuver, since most of the time the satellites acquire angles $\beta$ close to 90 deg, the orbital decay is not so significant. Finally, the resulting difference in the right ascension of the ascending nodes $\Delta \bar{\Omega}$ and the distance between the two satellites is also depicted.

Figures 3 and 4 illustrate that, using the proposed controller, long-range re-phasing maneuvers can be accomplished with DD, even in presence of uncertainties in the model of atmospheric density and drag coefficient.

## Short Range Rendezvous Maneuver

A simulation using the control law proposed in Eq. (75) was performed with the following initial conditions of the Chaser satellite with respect to the Target, resolved in the corresponding LVLH frame $\boldsymbol{v}_{0}=\left[\begin{array}{llll}-0.078 \mathrm{~km} & -28.589 \mathrm{~km} & -0.161 \cdot 10^{-4} \mathrm{~km} / \mathrm{s} & 0.396 \cdot 10^{-4} \mathrm{~km} / \mathrm{s}\end{array}\right]$. The initial inertial position and velocity for the Target were set as $\left[\begin{array}{llll}-225.874 & 2577.431 & -6271.345\end{array}\right]^{\top} \mathrm{km}$ and $\left[\begin{array}{lll}3.026 & -6.475 & -2.770\end{array}\right]^{\top} \mathrm{km} / \mathrm{s}$ respectively. These conditions were set according to the final conditions of the long-range re-phasing simulation. The dynamics of the Target were numerically integrated including two body forces, $J_{2}$ perturbation and the drag force with uncertainties (Eq. (81)), while the dynamics of the Chaser relative to the Target were modeled using the Schweighart and Sedwick model with uncertainties in the input (relative drag acceleration) (Eq. (64)), steered by the min-max control (Eq. (75)). The dynamics were simulated for 15 days.

The eigenvalues of the matrix $\overline{\mathbf{A}}$ were set to be: $\left[\begin{array}{lll}-3 \cdot 10^{-6} & -3 \cdot 10^{-7} & -3 \cdot 10^{-7}+1.130 \text {. }\end{array}\right.$ $10^{3} \cdot j-3 \cdot 10^{-7}+1.130 \cdot 10^{3} \cdot j$ ], which are the Eigenvalues of matrix $\mathbf{A}_{m}$ shifted by negative real numbers (the real parts of the set poles). It is important to note that saturation was not taken into account for the development of the min-max control. To address this issue the poles of matrix $\overline{\mathbf{A}}$ were chosen such that $\mathbf{K}_{P} \boldsymbol{v}_{0}$ (where $\boldsymbol{v}_{0}$ is the initial condition) would not result in saturation. Furthermore, the value of $\bar{\eta}_{w}$ will also provide some insight on the maximum area required for applying the min-max control without saturating the system.

In this simulation, the error in the density was modeled as

$$
\begin{equation*}
\delta \rho_{C / T}=0.1 \rho_{C / T_{\text {model }}}\left|\sin \left(\bar{n}_{C / T} t\right)\right| \tag{82}
\end{equation*}
$$

where $\bar{n}_{C / T}=\sqrt{\mu / \bar{a}_{C / T}^{3}}$. This value is bounded by $0.1 \rho_{C / T \text { model }}$, which is used to calculate $\bar{\eta}_{w}$ at each time step, using the current values of $\rho_{C / T}$ in Eq. (82).

Figure 5 (top three plots) shows the norms of the relative position (inter-spacecraft separation, first plot) and velocity (second plot), and the Lyapunov function (defined in Eq. (79), third plot), for applying the control law defined in Eq. (75), with and without the min-max. In the case without the min-max term, this control law reduces to the term $\mathbf{K}_{P} \boldsymbol{v}$, which is a linear state feedback control. The Lyapunov function for both cases is always decreasing; however, it decreases at a much faster rate for the case with the min-max term. It should be noticed that for the case without the min-max term, the fact that $\dot{V}_{m}$ is negative definite cannot be guaranteed. The inter-spacecraft separation approaches zero faster in the case without the min-max control, but then it overshoots significantly $(\sim 60 \mathrm{~km})$. On the contrary, the inter-spacecraft separation for the case with the min-max term
approaches zero at a slower rate. It still slightly overshoots but comes back towards the origin and remains in its vicinity. The resulting trajectory in the $x-y$ plane for the case with the min-max term is shown in Figure 5 (Bottom plot).


Figure 5. [Top three]Comparison between the simulations with and without the minmax control for the short range maneuver. [Bottom] Trajectory in the $x-y$ plane for the short range maneuver using the min-max control

The angles for both spacecraft for this simulation are shown in Figure 6 [Top], calculated using Eq. (77). This figure shows that the $\beta$ angles for both spacecraft remain close to $90^{\circ}$ during the whole maneuver. This indicates that the areas required by the min-max control remain close to the min value $S_{B}$. Consequently, at least during this maneuver the controller never saturates the system and the drag force remains low, which is desired to reduce the orbital decay caused by the maneuver.

Finally, Figure 6 [Bottom] shows the mean semi-major axis for the Target during the short range maneuver. This figure indicates that the orbital decay for the maneuver is very small ( $\sim 1.9 \mathrm{~km}$ ), and consequently the values $a$, and $b$, and matrix $\mathbf{A}_{m}$, change very little ( $\sim-0.042 \%$ and $\sim-0.084$ $\%$ for $a$ and $b$, respectively). This supports that the assumption of the constant $\mathbf{A}_{m}$ is valid for this type of maneuvers.

## CONCLUSIONS

This work presented a methodology to perform DD relative maneuvering of coplanar spacecraft, towards rendezvous, under bounded uncertainties in the drag acceleration. The required differential drag accelerations were obtained by varying the pitch angles of the satellites.

One approach enables to consider long-range maneuvers, assuming that both satellite are initially in circular orbits. The developed dynamical system, based on mean semi-major axes and mean arguments of latitude, allows for the implementation of an LQR controller with a saturation function. In presence of bounded uncertainties, convergence of the trajectories can be proved up to a certain norm of the state vector, for which an analytical expression was provided in terms of the uncertainties. The saturated configuration was examined by plotting the phase portrait of the system, which


Figure 6. [Top]Actuation for both spacecraft during the short range maneuver using the min-max control. [Bottom]Decay of the $\bar{a}_{T}$.
showed that the system will eventually reach the non-saturated configuration, and hence converge.
A second approach, implements a min-max controller on a linearized system that models the in-plane dynamics. The implemented controller provides asymptotic stability, provided that the uncertainties in the drag acceleration are bounded. The formulation allows to utilize this approach for shorter relative distances.

In general, the bounds on the uncertainties must be carefully determined by the control designers in a manner that they reflect their knowledge of error in the relevant parameters, without being excessively conservative. Highly conservative uncertainties would result in high actuation requirements which would increase the orbital decay and may result in saturation of the controllers.

## REFERENCES

[1] D. A. Vallado, Fundamentals of Astrodynamics and Applications. McGraw-Hill Inc., 2007.
[2] D. Drob, J. Emmert, G. Crowley, J. Picone, G. Shepherd, W. Skinner, P. Hays, R. Niciejewski, M. Larsen, C. She, et al., "An empirical model of the Earth's horizontal wind fields: HWM07," Journal of Geophysical Research: Space Physics (1978-2012), Vol. 113, No. A12, 2008.
[3] T. Finley, D. Rose, K. Nave, W. Wells, J. Redfern, R. Rose, and C. Ruf, "Techniques for LEO Constellation Deployment and Phasing Utilizing Differential Aerodynamic Drag," AIAA Astrodynamics Conference, 2013.
[4] B. S. Kumar and A. Ng, "A Bang-Bang Control Approach to Maneuver Spacecraft in a Formation with Differential Drag," Proceedings of the AIAA Guidance, Navigation and Control Conference and Exhibit, 2008, http://dx.doi.org/10.2514/6.2008-6469.
[5] C. Leonard, W. Hollister, and E. Bergmann, "Orbital formationkeeping with differential drag," Journal of Guidance, Control, and Dynamics, Vol. 12, No. 1, 1989, pp. 108-113, http://dx.doi.org/10.2514/3.20374.
[6] W. H. Clohessy and R. S. Wiltshire, "Terminal guidance system for satellite rendezvous," Journal of the Aerospace Sciences, Vol. 27, No. 9, 1960, pp. 653-658.
[7] T. Carter and M. Humi, "Clohessy-Wiltshire Equations Modified to Include Quadratic Drag," Journal of Guidance, Control, and Dynamics, Vol. 25, No. 6, 2002, pp. 1058-1063, http://dx.doi.org/10.2514/2.5010.
[8] R. Bevilacqua and M. Romano, "Rendezvous Maneuvers of Multiple Spacecraft using Differential Drag under J2 Perturbation," Journal of Guidance, Control, and Dynamics, Vol. 31, No. 6, 2008, pp. 15951607, http://dx.doi.org/10.2514/1.36362.
[9] R. Bevilacqua, J. S. Hall, and M. Romano, "Multiple spacecraft rendezvous maneuvers by differential drag and low thrust engines," Celestial Mechanics and Dynamical Astronomy, Vol. 106, No. 1, 2010, pp. 69-88, http://dx.doi.org/10.1007/s10569-009-9240-3.
[10] S. A. Schweighart and R. J. Sedwick, "High-Fidelity Linearized J Model for Satellite Formation Flight," Journal of Guidance, Control, and Dynamics, Vol. 25, No. 6, 2002, pp. 1073-1080, http://dx.doi.org/10.2514/2.4986.
[11] D. Pérez and R. Bevilacqua, "Differential drag spacecraft rendezvous using an adaptive Lyapunov control strategy," Acta Astronautica, Vol. 83, 2013, pp. 196-207, http://dx.doi.org/10.1016/j.actaastro.2012.09.005.
[12] O. Ben-Yaacov and P. Gurfil, "Long-Term Cluster Flight of Multiple Satellites using Differential Drag," Journal of Guidance, Control, and Dynamics, Vol. 36, No. 6, 2013, pp. 1731-1740, http://dx.doi.org/10.2514/1.61496.
[13] L. Dell'Elce and G. Kerschen, "Comparison between analytical and optimal control techniques in the differential drag based rendez-vous," Proceedings of the 5th International Conference on Spacecraft Formation Flying Missions \& Technologies, 2013.
[14] C. Tuttle, "Satellite Stationkeeping of the ORBCOMM Constellation Via Active Control of Atmospheric Drag: Operations, Constraints, and Performance (AAS 05-152)," 2005.
[15] D. A. Vallado and D. Finkleman, "A critical assessment of satellite drag and atmospheric density modeling," Acta Astronautica, Vol. 95, 2014, pp. 141-165, http://dx.doi.org/10.1016/j.actaastro.2013.10.005.
[16] S. Gutman, "Uncertain dynamical systems-A Lyapunov min-max approach," Automatic Control, IEEE Transactions on, Vol. 24, No. 3, 1979, pp. 437-443, http://dx.doi.org/10.1109/TAC.1979.1102073.
[17] D. Mishne, "Formation Control of Satellites Subject to Drag Variations and $J_{2}$ Perturbations," Journal of Guidance, Control and Dynamics, Vol. 27, July-August 2004, pp. 685-692, http://dx.doi.org/10.2514/1.11156.
[18] H. Schaub, S. R. Vadali, J. L. Junkins, and K. T. Alfriend, "Spacecraft Formation Flying Control using Mean Orbit Elements," Journal of the Astronautical Sciences, Vol. 48, No. 1, 2000, pp. 69-87.
[19] H. Schaub and K. T. Alfriend, "Impulsive Feedback Control to Establish Specific Mean Orbit Elements of Spacecraft Formations," Journal of Guidance, Control, and Dynamics, Vol. 24, No. 4, 2001, pp. 739745.
[20] L. Mazal and P. Gurfil, "Closed-Loop Distance-Keeping for Long-Term Satellite Cluster Flight," Acta Astronautica, Vol. 94, No. 1, 2014, pp. 73-82, http://dx.doi.org/10.1016/j.actaastro.2013.08.002.
[21] R. Battin, An Introduction to the Mathematics and Methods of Astrodynamics. AIAA Education Series, 1999, http://dx.doi.org/10.2514/4.861543.
[22] H. Kwakernaak and R. Sivan, Linear optimal control systems, Vol. 1. Wiley-interscience New York, 1972.
[23] H. K. Khalil, Nonlinear systems. Prentice Hall, New Jersey, 2000.
[24] K. Alfriend, S. Vadali, P. Gurfil, J. How, and L. Breger, Spacecraft Formation Flying: Dynamics, Control and Navigation. Elseiver, Oxford, UK, 2010.
[25] P. M. Mehta, A. Walker, C. A. McLaughlin, and J. Koller, "Comparing Physical Drag Coefficients Computed Using Different Gas-Surface Interaction Models," Journal of Spacecraft and Rockets, Vol. 51, No. 3, 2014, pp. 873-883.
[26] S. Bruinsma, "DTM2013 Evaluation Report," tech. rep., CNES, 2013. http://www.atmop.eu/publicdocuments/evaluation_report_dtm2013.pdf.
[27] H. Schaub and J. Junkins, Analytical Mechanics of Aerospace systems, Vol. 1. AIAA, 2003, Appendix F, pp. 693-696.


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[^1]:    *In fact, drag reduces the eccentricity of orbits. [1, Page 671]
    ${ }^{\dagger}$ This approximation has been proposed and assessed in a few papers ${ }^{17-20}$ showing its validity.

[^2]:    ${ }^{*}$ The same rationale can be applied if density estimation is considered. Nevertheless, nowadays real-time density estimation is not a trivial task.
    ${ }^{\dagger}$ Notice that for the forthcoming analysis, the state vector as well as any other quantity involved can be considered as normalized by corresponding units; i.e. distances by km, angles by rad, and time by seconds.

[^3]:    *Any density variations due to solar activity would result in time-varying coefficients for the model (44). Yet, the same relation (45) would be obtained, provided that the semi-major axes of the satellites are sufficiently close.

