# A SEARCH FOR INVARIANT RELATIVE SATELLITE MOTION 

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#### Abstract

Relative motion between two or more satellites in formation has been studied for quite a long time, as the works of W.H. Clohessy and R.S. Wiltshire, dated 1960, or the studies of J. Tschauner, dated 1967, can demonstrate. These early works not only are milestones for the whole future research, as they provide linear models which ensure optimum performances in terms of motion prediction in the simplified assumption of pure Keplerian motion, but are also powerful means for phenomenon comprehension. In fact these models furnish conditions on the initial relative position and velocity so that the relative orbits result to be periodic, that is closed orbits. When perturbations, such as $\mathrm{J}_{2}$ and drag effects, or simple nonlinearities are taking into account into the model, the analytical solution appears harder and harder to be found, if not impossible. Simple relations for periodic orbits such as in Hill-Clohessy-Wiltshire (HCW) equations are not to be expected. A more suitable approach seems to be the numerical one. The research here presented aims finding initial conditions for bounded orbits in the case of a nonlinear model through the use of a Genetic Algorithm. Before using the GA for the nonlinear problem, the optimization method is tested on the Hill's and Tschauner-Hempel's models, where the analytical solution is well known. The algorithm runs considering the initial relative velocities between the satellites as the individuals of the population, leaving unchanged the initial relative position. This not only reduces the number of variables the GA is working with, but means searching closed relative orbits of a pre-fixed dimension. Using the numerical results for a 0.3 eccentric orbit, the equality condition between the two satellites' semi-major axis has been re-obtained; the initial velocities generated with the GA result to better fit the requirements of orbit closing than the analytical T.-H. conditions as the orbit dimension grows.

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## Nomenclature

LVLH = Local Horizontal Local Vertical
$x, y, z=$ relative position in LVLH frame
$\dot{x}, \dot{y}, \dot{z}=$ relative velocity in LVLH frame
subscript $i=$ values at the initial time
subscript $f=$ values at the final time
$f=$ fitness function
$\omega_{0}=$ angular velocity of the circular orbit
$a=$ semi-major axis
$e=$ eccentricity
$i=$ inclination
$\Omega=$ RAAN
$\omega=$ argument of perigee
$n=$ mean motion

## INTRODUCTION

Many efforts have been made in the last years on modeling and control of satellites formations. In (Schaub, 1999, [7]) conditions for relative orbits invariant with respect to the $\mathrm{J}_{2}$ perturbation are presented in terms of mean orbital elements. In literature several linear models of relative dynamics including the second harmonic of the gravitational field, eccentricity an the air drag can be found (Izzo, 2002, [4]; Sabatini, 2003, [6]; Tschauner, 1967, [8]) but nor the analytical
solution neither the initial conditions for periodic relative orbits are obtainable in the most of the cases.
The use of evolutionary/genetic approaches in the aerospace research, especially in mission analysis and study phase, is quite recent (Kim, 2002, [5]). The difficulties encountered in a genetic optimum search stand in the strong dependence they show on the fitness function choice, mutation and crossover probabilities, the population size and the number of generations. There is not a rigorous mathematical way to choose them in the proper manner and the convergence can be achieved only after a trial and error adjustment of the parameters with respect to the particular problem. Anyway GA are useful to find global optimum for problems that admit more then one local maximum, and in which the behavior of the function to be maximized is unknown.
The new interest towards genetic approaches in this field of research can be found as well in the proposals for academic studies financed by the European Space Agency (ESA, 2004, [2]).

## GENETIC ALGORITHMS

GA are stochastic global search methods that are based on the principle of natural selection and evolution of the species. This kind of algorithms results to be effective for optimization problems containing different local optima with discontinuous parts between them. In these cases the calculus-based models can converge to a local optimum which is not the absolute optimum.
In the present paper the GA is first validated for the formation flying problem restricted to the Hill's and Tschauner-Hempel's linearized models. The first class of dynamics considers a reference orbit without eccentricity, the second includes it in the keplerian linearized motion. After the numerical proof for the convergence to the well-known conditions for closed orbits, the method is run for the nonlinear keplerian relative motion in presence of a 0.3 eccentricity value. The relative motion is in the LVLH frame.
The fitness function chosen is here reported:

$$
f(x, y, z)=-\sqrt{\left(x_{f}-x_{i}\right)^{2}+\left(y_{f}-y_{i}\right)^{2}+\left(z_{f}-z_{i}\right)^{2}}
$$

representing the error in relative position between the initial conditions and the ones obtained at the end of the integration. The integration is performed on one orbital period for Hill and T.-H.. For the nonlinear model 5 periods have been used to make the algorithm converge in a satisfying manner.

The reason for the genetic choice stands in the not known form of the function to be optimized in the nonlinear problem. Instead, in the Hill and T.-H. models, the fitness function is expected to have a parabolic-like behavior reported in the ( $e, \dot{y}$, fitness value) space.
In all the mentioned runs the initial relative position is fixed and the relative velocities are the only variables of the GA.
The software used for the numerical search is the online PIKAIA freely available tool (Charbonneau, 1995, [1]). PIKAIA uses a decimal alphabet made of 10 simple integers (0 through 9) for encoding the chromosome ( $\dot{x}, \dot{y}, \dot{z}$ ). The mutation and crossover characteristic are the default PIKAIA's ones (see Charbonneau, 1995, [1]).

## VALIDATION OF THE GA

## Using genetic algorithm to find analytical Hill's solutions

Here is reported the analytical condition for the Hill's dynamic to be closed for an orbit with the following angular velocity:
$\frac{\dot{y}}{x}=-2 \omega_{0}=-0.002156 \mathrm{rad} / \mathrm{s}$
The simulations reported in Table 1 show how the GA is able to obtain the desired solution setting by trial and error the parameters:

| Individuals | Generations | $\dot{y} / x$ pikaia | Fitness function |
| :---: | :---: | :---: | :---: |
| 20 | 50 | $-2.164199 \mathrm{E}-03$ | $-2.954054 \mathrm{E}-02$ |
| 20 | 100 | $-2.153799 \mathrm{E}-03$ | $-6.665711 \mathrm{E}-04$ |
| 50 | 100 | $-2.153799 \mathrm{E}-03$ | $-5.935614 \mathrm{E}-04$ |
| 100 | 100 | $-2.153998 \mathrm{E}-03$ | $-4.884340 \mathrm{E}-04$ |
| 100 | 500 | $-2.154398 \mathrm{E}-03$ | $-4.270749 \mathrm{E}-04$ |
| 100 | 1000 | $-2.154398 \mathrm{E}-03$ | $-4.260100 \mathrm{E}-04$ |

Table 1: convergence of the GA increasing generations and population size

## Using genetic algorithm to find analytical Tschauner-Hempel's solutions

Here is reported the analytical condition for the Tschauner-Hempel's dynamic to be closed (Inalhan, 2002, [3]):

$$
\frac{\dot{y}(t=0)}{x(t=0)}=-\frac{n(2+e)}{(1-e)^{\frac{3}{2}}(1+e)^{\frac{1}{2}}}
$$

## Equation 1: closed orbit condition for T.-H.

To perform the analysis 100 individuals and 100 generations have been used. Values higher than these ones in the Hill search demonstrated to not improve the quality of the solution.

| Eccentricity | $\dot{y} / x$ analytical | $\dot{y} / x$ pikaia | error | Fitness function |
| :---: | :---: | :---: | :---: | :---: |
| $e=0$ | -2 | -1.999999 | $1 \mathrm{e}-7$ | $-3.4 \mathrm{e}-10$ |
| $e=0.1$ | -1.9091 | -1.90899 | $1.1 \mathrm{e}-4$ | $-5.2 \mathrm{e}-8$ |
| $e=0.2$ | -1.8333 | -1.83339 | $9 \mathrm{e}-5$ | $-6.9 \mathrm{e}-8$ |
| $e=0.3$ | -1.7692 | -1.769199 | $1 \mathrm{e}-6$ | $-2.9 \mathrm{e}-8$ |
| $e=0.4$ | -1.7143 | -1.714199 | $1.01 \mathrm{e}-4$ | $-8.9 \mathrm{e}-7$ |
| $e=0.5$ | -1.6667 | -1.666599 | $1.01 \mathrm{e}-4$ | $-1.2 \mathrm{e}-6$ |
| $e=0.6$ | -1.625 | -1.624998 | $2 \mathrm{e}-6$ | $-3.3 \mathrm{e}-10$ |
| $e=0.7$ | -1.58823 | -1.58819 | $1.4 \mathrm{e}-4$ | $-4.9 \mathrm{e}-6$ |
| $e=0.8$ | -1.55556 | -1.555599 | $3.9 \mathrm{e}-5$ | $-6.4 \mathrm{e}-5$ |
| $e=0.9$ | -1.526315 | -1.526399 | $8.4 \mathrm{e}-5$ | $-1.6 \mathrm{e}-5$ |

Table 2: GA restituting the analytical T.-H. conditions for different eccentricities

The results in Table 2 are plotted in Figure 1:


Figure 1: TH analytical solution vs TH solution with GA

As it was expected the behavior of the fitness function indicates an infinite set of minimum but located in according to the Equation 1 (real fitness has to be changed in sign with respect to the one here reported; the minimums in the graphic are optima maximum for the GA):


Figure 2: fitness value vs eccentricity and $\dot{y}$

## SEARCHING CLOSED RELATIVE ORBITS FOR THE NONLINEAR MODEL

The integration of the dynamic in the nonlinear problem arises difficulties for the calculation time and the accuracy of the solution. Subtracting directly the cartesian coordinates of the two satellites can easily degrade the quality of the relative position obtained, working with very close values. In (Vadali, 2002, [9]) an approach based on a geometric method (called unit sphere projection) is proposed. Integrating the relative dynamic in terms of orbital elements (for the keplerian case just the true anomaly has to be used, see [9]) and subsequently translate the differences in terms $\delta x, \delta y, \delta z$ is numerically more accurate and the computation time is dramatically reduced. This approach has been here used. The genetic parameters are:

| Crossover probability | Mutation rate |  |  |
| :---: | :---: | :---: | :---: |
|  | initial | minimum | maximum |
| 0.85 | 0.005 | 0.0005 | 0.25 |

Table 3: genetic parameters for nonlinear approach
After numerous trials the number of generations has been set to 500 with a population of 100 individuals and simulations have been performed for different relative orbit sizes. The initial dimension of these orbits increase from a 2 km relative position on the three axes to a 500 km one. Comparing the analytical relation of T.-H. with the $\dot{y} / x$ rate obtained trough the GA it can be noted how the linear condition looses its validity as the dimensions increase:


Figure 3: $\frac{\dot{y}}{x}$ rate compared for the linearized and the nonlinear models (logarithmic scale on $x$ axis)
Figure 3 shows the matching between the T.-H. approach and the complete one for relative orbits of low dimensions. As the size increases the rate for the nonlinear model goes down loosing the constant behavior. As expected the T.-H. relation permits the orbit closing just for modest values for the initial position. Then the GA results to better fit the requirement of bounded motion. Here are illustrated 10 orbits obtained with the T.-H. and GA for a low distance initial value ( 2 km ) ad a higher one ( 200 km ):


Figure 4: 10 orbits (T.-H. vs nonlinear) for low size ( 2 km )


Figure 5: 10 orbits (T.-H. vs nonlinear) for high size (200 km)

As an additional proof of the GA convergence to a satisfying solution the orbital parameters of the two spacecrafts are calculated and compared:

| Vehicle \#1 |  | Vehicle \#2 |  | \% difference |
| :---: | :---: | :---: | :---: | :---: |
| $a$ | 7000 km | $a$ | 7000.124 km | $0.0018 \%$ |
| $e$ | 0.3 | $e$ | 0.2928 | $2.4 \%$ |
| $i$ | 35 deg | $i$ | 35.012 deg | $0.034 \%$ |
| $\Omega$ | 35 deg | $\Omega$ | 33.99 deg | $2.9 \%$ |
| $\omega$ | 35 deg | $\omega$ | 1.405 deg | $95.99 \%$ |

The only parameter that clearly maintains its value unchanged (considering the numerical errors) is the semi-major axis $a$. This results coincide with the only constrain to close a relative orbit in a keplerian motion: the equality of the semi-major axis. In this way the two orbits have the same orbital period T and obviously the relative position is repeated every T seconds.

## CONCLUSIONS

The GA strategy here used resulted to be a valid instrument to analyze the behaviour of the nonlinear relative dynamics between two satellites in keplerian orbit. After having re-obtained the Hill's and T.-H.'s solutions for bounded trajectories to check the validity of the algorithm, the GA has been run for the complete mathematical model of relative motion in keplerian orbit.
Considering the numerical approach and the limitations in terms of accuracy for the solutions, the matching period condition have been obtained for closing the relative orbit. The initial velocities generated with the genetic calculation match the analytic relation for T.-H. demonstrating the validity of the linear approach for low dimensions orbits. Increasing size results in a obliged switching to the conditions obtained numerically.

Future developments of this new approach to the formation flying problem include the analysis of $\mathrm{J}_{2}$ and drag effects. The present paper represents an introduction and a validation work for the authors whose aim is to apply and study the possibilities given by the genetic algorithm to the most complete as possible model of the relative dynamics of satellites.

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