Optimal large reorientation manoeuvre of a spinning gyrostat¹

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Abstract

The optimal control problem of the spin axis of a gyrostat is studied and solved via a direct transcription method. Dynamical equations suitable for the problem considered are presented together with Pontryagin necessary conditions. The optimal control problem is then transcribed into a highly sparse non linear programming problem. A technique is presented to generate first guess solutions that are suitable for various objective functions. Time optimal control is then compared to a mixed minimum time minimum control effort strategy in a particular case. The technique, relevant to New Earth Object deflection strategies, is then applied to ESA Pluto Orbiter Probe concept study and optimal and sub-optimal solutions are compared.

Introduction

Axis control of spinning bodies has recently become a very relevant issue in the optimal deflection of Near Earth Objects (NEOs). A number of concepts have been developed, that require some control on the asteroid spin axis in order to optimally orient the thrust provided by an engine placed in the pole. Large spinning structures, such as those that might serve as interplanetary gateways (not at all science fiction as the recent improvements [7] on the three body problem have shown), would also be affected by the spin axis control capabilities of the station. A more classical example in which spin axis control is needed is, of course, that of the attitude control of spinning spacecrafts (many interplanetary missions have a spinning phase that provide accurate orientation during engines burns as well as other benefits), even if in this case optimality is rarely an issue. In mathematical terms, the problem of optimal attitude control does not admit an analytical solution if not in simplified cases. One of these is that of a gyroscope, when the sole component of the angular velocity perpendicular to the spin axis is controlled. In the vast majority of the practical problems no analytical solution is available and one has to try solving the problem numerically. A fairly good number of computational techniques [8] already exist that

¹Professor Chiara Valente died before she could see the final release of this work . We would like to dedicate our work in this paper to her, being a little thank to her unvaluable support.

solve optimal control problems. Direct transcription methods, based on transforming the continuous optimal problem into a Non Linear Programming problem, are surely quite promising, being the most simple and powerful ones. In fact they are able to solve all kind of optimal problems with the possibility of implementing both state and control constrains and both equality and inequality constrains. The computational efficiency of the Non Linear Programming solver is, of course, of the greatest importance as well as the accuracy of the first guess solution. This is why particular attention is here paid to these issues.

Model choice and problem statement

It has been widely recognized that quaternion algebra is an invaluable tool to efficiently write the vast majority of attitude dynamic related softwares. Singular sets of parameters, though, may still have some advantages over the quaternion. Such is the case of the Lyapunov based control of Euler Equations that may be done in complicated cases such as that of spacecrafts mounting new generation fly-wheels, with the aid of the Modified Rodriguez Parameters (singular for rotations of π) (see for example [6, 5]). In this work we make use of a rather classical set of singular parameters defining the spacecraft attitude, the Yaw Pitch and Roll angles, also called Euler 321 angles. For an exhaustive definition of these angles see Hughes [1]. The great advantage of this choice, in developing the optimization algorithm for the reorientation of a spinning gyrostat, stems from the fact that it makes easier to write the final desired conditions in terms of the state. This may be shown with the following simple calculation. Let us find the desired final quaternion in a spin axis reorientation manoeuvre. This is a case in which there is not a single value of the quaternion to be targeted but a whole set, as a degree of freedom is left free (i.e. the rotation about the final spin axes). The set of all quaternions $[\varepsilon_1, \varepsilon_2, \varepsilon_3, \eta]$ describing the desired final state may then be evaluated by observing that it will be generated by integrating the kinematic equations:

$$\begin{array}{l} 2\dot{\epsilon}_1=\Omega\eta\\ 2\dot{\epsilon}_2=\Omega\epsilon_3\\ 2\dot{\epsilon}_3=-\Omega\epsilon_2\\ 2\dot{\eta}=-\Omega\epsilon_1 \end{array}$$

with initial conditions:

$$\epsilon_1 = 0$$
 $\epsilon_2 = 0$ $\epsilon_3 = \sin \frac{\Psi}{2}$ $\eta = \cos \frac{\Psi}{2}$

being Ω the spin velocity of the spacecraft and $\hat{\Psi}$ the amplitude of the reorientation. We get the solution:

$$\begin{aligned} \varepsilon_1 &= \cos\frac{\Psi}{2}\sin\frac{\Omega}{2}t\\ \varepsilon_2 &= \sin\frac{\Psi}{2}\sin\frac{\Omega}{2}t\\ \varepsilon_3 &= \sin\frac{\Psi}{2}\cos\frac{\Omega}{2}t\\ \eta &= \cos\frac{\Psi}{2}\cos\frac{\Omega}{2}t \end{aligned}$$

The final set of the desired quaternions may therefore be defined as follows:

$$\begin{aligned} \varepsilon_{1_f}^2 + \eta_f^2 &= \cos^2 \frac{\Psi}{2} \\ \varepsilon_{2_f}^2 + \varepsilon_{3_f}^2 &= \sin^2 \frac{\hat{\Psi}}{2} \\ \varepsilon_{1_f} \varepsilon_{2_f} + \varepsilon_{3_f} \eta_f &= \frac{1}{2} \sin \hat{\Psi} \end{aligned}$$

On the other hand, the 321 Euler angles (see figure 1) defined as:

$$\Psi$$
: precession angle (yaw)
 μ : nutation angle (pitch)
 φ : rotation angle (roll)

lead to a final state defined by:

$$\Psi_f = \hat{\Psi}$$
$$\mu_f = 0$$
$$\varphi_f = \text{any}$$

From a numerical point of view the advantages of using Euler Angles for our purposes are then obvious. We are able to easily write the final state and to use three kinematic equations rather than four. Besides the singularity problem of Euler Angles will not be encountered as the nutation angle is unlikely to grow high. If the component of a



Figure 1: Euler angles used

vector $\vec{\mathbf{v}}$ are known in the inertial frame \mathcal{F}_i then its component in the body frame \mathcal{F}_b will be $\mathbf{v}_b = \mathbf{C}_{bi} \mathbf{v}_i$. The rotation matrix \mathbf{C}_{bi} has the following expression:

$$\mathbf{C}_{bi} = \begin{bmatrix} \cos\mu\cos\psi & \cos\mu\sin\psi & -\sin\mu\\ \sin\phi\sin\mu\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\mu\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\mu\\ \cos\phi\sin\mu\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\mu\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\mu \end{bmatrix}$$

This rotation matrix is part of the dynamic equations whenever the inertial components of the torque are considered as controls rather than the body axes components. The complete set of differential equations that is here considered to solve the optimal spin axis reorientation problem is:

$$\begin{cases} \mathbf{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} = \mathbf{u}_b = \mathbf{C}_{bi}\mathbf{u}_i \\ \dot{\boldsymbol{\psi}} = \frac{1}{\cos\mu}(q\sin\varphi + r\cos\varphi) \\ \dot{\boldsymbol{\mu}} = q\cos\varphi - r\sin\varphi \\ \dot{\boldsymbol{\varphi}} = \frac{\sin\mu}{\cos\mu}(q\sin\varphi + r\cos\varphi) + p \end{cases}$$

that, for a gyroscopic satellite become:

$$\begin{cases} \dot{p} = \frac{1}{A}u_{b_x} \\ \dot{q} = \frac{1}{B}[(B-A)pr + u_{b_y}] \\ \dot{r} = \frac{1}{B}[(A-B)pq + u_{b_z}] \\ \dot{\psi} = \frac{1}{\cos\mu}(q\sin\varphi + r\cos\varphi) \\ \dot{\mu} = q\cos\varphi - r\sin\varphi \\ \dot{\varphi} = \frac{\sin\mu}{\cos\mu}(q\sin\varphi + r\cos\varphi) + p \end{cases}$$

being *A*,*B*,*C* the three principal inertia moments with B = C. We will consider as state vector $\mathbf{x} = (p,q,r, \psi, \mu, \varphi)^T$ and as control vector $\mathbf{u}_i = (u_{ix}, u_{iy}, u_{iz})^T$ or $\mathbf{u}_b = (u_{bx}, u_{by}, u_{bz})^T$. Having chosen the dynamic we may now define our problem, that in the Bolza form, has the form:

$$\min_{\mathbf{u}_b = \mathbf{C}_{bi} \mathbf{u}_i \in \mathbf{U}} J[\mathbf{x}(t), \mathbf{u}(t), t] = \varphi[\mathbf{x}(t_f), \mathbf{u}(t_f), t_f] + \int_{t_0}^{t_f} \mathcal{L}[\mathbf{x}(t), \mathbf{u}(t), t] dt$$

subject to the following dynamical constraints and border conditions:

$$\begin{array}{ll} p(0) = \Omega & \psi(0) = 0 & p(t_f) = \Omega & \psi(t_f) = \hat{\Psi} \\ q(0) = 0 & , \ \mu(0) = 0 & , \ q(t_f) = 0 & , \ \mu(t_f) = 0 \\ r(0) = 0 & \phi(0) = 0 & r(t_f) = 0 & \phi(t_f) = \text{any} \end{array} ,$$

The domain $\mathbf{U}\in R^3$ is typically a cube or a cuboid, but may be of a more general form.

In order to show the difficulties we would encounter trying an analytical approach, we briefly write the necessary optimality conditions given by the Pontryagin maximum principle, see ([4]). In the simplest case of minimum time optimal control the Hamiltonian H is:

$$H = \psi_0 + \psi_1[\frac{1}{A}(u_{b_x})] + \psi_2[\frac{1}{B}((B-A)pr + u_{b_y})] + \psi_3[\frac{1}{B}((A-B)pq + u_{b_z})] + \psi_4[\frac{1}{\cos\mu}(q\sin\phi + r\cos\phi)] + \psi_5[q\cos\phi - r\sin\phi] + \psi_6[\frac{\sin\mu}{\cos\mu}(q\sin\phi + r\cos\phi) + p]$$

where ψ_i are the auxiliary functions. The complete Differential Algebraic Equations

we have to solve are therefore:

$$\begin{cases} \dot{p} = \frac{1}{A}(u_{b_x}) \\ \dot{q} = \frac{1}{B}((B-A)pr + u_{b_y}) \\ \dot{r} = \frac{1}{B}((A-B)pq + u_{b_z}) \\ \dot{\psi} = \frac{1}{\cos\mu}(q\sin\phi + r\cos\phi) \\ \dot{\mu} = q\cos\phi - r\sin\phi \\ \dot{\phi} = \frac{\sin\mu}{\cos\mu}(q\sin\phi + r\cos\phi) + p \\ \dot{\psi}_1 = \psi_2[\frac{1}{B}(B-A)r] + \psi_3[\frac{1}{B}(A-B)q] + \psi_6\frac{\sin(\mu)}{\cos(\mu)} \\ \dot{\psi}_2 = \psi_3[\frac{1}{B}(A-B)p] + \psi_4\frac{1}{\cos(\mu)}\sin(\phi) + \psi_5\cos(\phi) + \psi_6\frac{\sin(\mu)}{\cos(\mu)}\sin(\phi) \\ \dot{\psi}_3 = \psi_2[\frac{1}{B}(B-A)p] + \psi_4\frac{1}{\cos(\mu)}\cos(\phi) - \psi_5\sin(\phi) + \psi_6\frac{\sin(\mu)}{\cos(\mu)}\cos(\phi) \\ \dot{\psi}_4 = 0 \\ \dot{\psi}_5 = \psi_4(q\sin(\phi) + r\cos(\phi))\frac{\sin(\mu)}{\cos^2(\mu)} + \psi_6(q\sin(\phi) + r\cos(\phi))\frac{1}{\cos^2(\mu)} \\ \dot{\psi}_6 = \psi_4[\frac{1}{\cos\mu}(q\cos\phi - r\sin\phi)] + \psi_5[-q\sin\phi - r\cos\phi] + \\ + \psi_6[\frac{\sin\mu}{\cos\mu}(q\cos\phi - r\sin\phi)] \\ max H, \forall t \end{cases}$$

subject to boundary conditions at the initial time and at the final time (transversality conditions have to accounted for in this case). Even in the simplest case, such as the one of an inertially symmetric satellite, the problem of minimum time brings to an analytically unsolvable system (see [2]).

Solving the optimal problem

We here describe the numerical method used to solve this problem. The method is a standard direct transcription method, see for example [9], excepts in the defects definition. Let the time scale be divided into N points t_k , k = 1..N. The variables considered in the Non Linear Programming (NLP) problem are:

$$\mathbf{z} = [\mathbf{x}_k, \mathbf{u}_k, \mathbf{u}_{mk}, \mathbf{x}_N, \mathbf{u}_N, t_f], \quad k = 1..N - 1$$

where \mathbf{x}_k is the state at time t_k , \mathbf{u}_k is the control at time t_k and u_{mk} is the control at time $\frac{t_k+t_{k+1}}{2}$. The continuous constrains of the Optimal Control Problem (OCP) have now to be transcribed into some algebraic constrains. This is done by using a fourth order Runge Kutta formula exploiting the states at the points t_k and the controls at the point t_k and $\frac{t_k+t_{k+1}}{2}$. The defects have been written as:

$$\boldsymbol{\zeta}_k = \mathbf{x}_{k+1} - \mathbf{x}_k - \frac{h_k}{6} [\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4]$$

where:

$$\mathbf{k}_1 = \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k)$$

$$\mathbf{k}_2 = \mathbf{f}(\mathbf{x}_k + \frac{\mathbf{k}_1}{2}, \mathbf{u}_{mk})$$

$$\mathbf{k}_3 = \mathbf{f}(\mathbf{x}_k + \frac{\mathbf{k}_2}{2}, \mathbf{u}_{mk})$$

$$\mathbf{k}_4 = \mathbf{f}(\mathbf{x}_k + \mathbf{k}_3, \mathbf{u}_{k+1})$$

This method, if compared to an Hemite-Simpson collocation, retains the NLP sparsity and increases the integration accuracy. To complete the transcription of the OCP into a NLP the objective function in the Bolza form has to be transcribed too with some quadrature formula. Newton-Cotes formula have here been implemented.

The initial guess

In the numerical solution of an optimal problem the initial guess plays a fundamental part. The initial guess used here is the following precession:

$$\begin{aligned} \Psi(t) &= \Psi(t) \\ \mu(t) &= 0 \\ \varphi(t) &= \Omega t \end{aligned}$$

where the unknown function $\Psi(t)$ is determined by means of variational calculus in the functional space defined by the constrains:

$$\Psi(0) = 0 \quad \dot{\Psi}(0) = 0 \quad \Psi(T) = \hat{\Psi} \quad \dot{\Psi}(T) = 0$$

Inverting the kinematic relations, the following holds for the body axis components of the angular velocity:

$$p(t) = \Omega$$

$$q(t) = \dot{\Psi}\sin\varphi$$

$$r(t) = \dot{\Psi}\cos\varphi$$

The corresponding control variables can be found by plugging the initial guess solution into the equation of motions, obtaining:

$$u_{ix} = -A\Omega \sin \Psi \dot{\Psi}$$
$$u_{iy} = A\Omega \cos \Psi \dot{\Psi}$$
$$u_{iz} = B \ddot{\Psi}$$

or,

$$u_{bx} = 0$$

$$u_{by} = B\sin\varphi\ddot{\Psi} + A\Omega\cos\Omega t\dot{\Psi}$$

$$u_{bz} = B\cos\varphi\ddot{\Psi} - A\Omega\sin\Omega t\dot{\Psi}$$

Note that if $\Omega = 0$, that is if the spacecraft is not spinning, we would have to apply a torque $B\Psi$ along the *z* axes in order to achieve the desired precession. The effect of the spin on our manoeuvre is, as we expected, all in the torque component perpendicular to the spin axis, and is directly proportional to the spin velocity. To find $\Psi(t)$ we consider the following variational problem: find Ψ that minimizes the control effort defined by the quantity:

$$P = \int_0^{t_f} \mathbf{u} \cdot \mathbf{u} dt = \int_0^{t_f} A^2 \Omega^2 \dot{\Psi}^2 + B^2 \dot{\Psi}^2 dt$$

Note that this will lead to the exact solution of an optimal control problem, that is the fixed time optimal reorientation of a gyrostat with one degree of freedom (precession angle) and with no limitations on the control. The solution to this problem may be obtained in closed form:

$$\Psi(t) = c_1 e^{\lambda t} + c_2 e^{-\lambda t} + c_3 t + c_4$$

where $\lambda = \frac{A}{B}\Omega$ and

$$c_{1} = -\frac{\Psi e^{-\lambda f}}{R}$$

$$c_{2} = \frac{\Psi}{R}$$

$$c_{3} = \frac{\Psi \lambda (1+e^{-\lambda f})}{R}$$

$$c_{4} = \frac{\Psi (e^{-\lambda f}-1)}{R}$$

$$R = 2e^{-\lambda f} + \lambda t_{f} (e^{-\lambda f}+1) - 2$$

In this problem the time t_f is considered fixed. In a practical optimization this is not an intersecting case and t_f has to be optimized as well. If, though, one tries to minimize the control effort, this would results in an infinite t_f . One has therefore to consider either a time minimal problem or, as suggested here, a mixed time-power optimization problem, in which case the following cost function can be used:

$$J(\mathbf{u},t_f) = t_f + k_0 \int_0^{t_f} \mathbf{u} \cdot \mathbf{u} dt$$

Having solved the fixed t_f optimal problem we may easily evaluate, for each t_f , the above cost function and therefore determine the optimal t_f (the fixed t_f optimal solutions represent Pareto-optimal solutions in the simple multi-objective optimization where both mass and time has to be minimized). We might also choose our t_f by taking care that the limits on the control values are satisfied (in this case J is not minimum, but the resulting initial guess may speed up the NLP solver convergence).

Results

Some results are shown for a selected case. A satellite with $A = 5Kgm^2$ and $B = 20Kgm^2$ is considered. The satellite is supposed to be 3-axis stabilized and the maximum torque around each of the axis is set to be 1Nm. The spin velocity Ω around the *x* inertia axis is supposed to be equal to 2.8 rpm and the final reorientation angle $\hat{\Psi} = \frac{\pi}{4}$. With these parameters a time optimal control is first obtained.

$$J(t_f) = t_f$$

In figure 2 the optimal control, both in body axis and in inertial axis, is shown together with the resulting state. The optimal control shows to be bang-bang in all the three components of the torque. This, as known, is not a general rule ([2], p.400) in attitude optimal control problems. The optimization returns a minimum time of $t_m = 7.82sec.$.



Figure 2: Optimal trajectory (minimum time)

The control effort, measured by the integral $P = \int_0^{t_f} \mathbf{u} \cdot \mathbf{u} dt$ has is $22.82N^2m^2sec$. The mixed time-power optimal problem is then considered, the result being shown in figure 3. The manoeuvre time is slightly increased, so that $t_f = 9.6sec$., but the control effort drops dramatically being $P = 3.27N^2m^2sec$. This result is quite general, as other simulations confirmed, leading to think to the consideration of this mixed problem as a good multi-objective optimization solution.

POP mission

The Pluto Orbiter Probe is a feasibility study performed by the ACT in ESTEC addressing the possibility of a Pluto exploration mission that may assess the nature of the so-called Pioneer Anomaly (see Rathke[3]) during its long (28 yeras) journey. In order to perform its mission, POP has to be spin stabilized (during a coast arc of 8.22 years) and it has to reorient its spin axis by an amplitude of the order of $\frac{\pi}{180}$ several times. As a consequence, its hollow-cathode thrusters, capable of exerting a maximum torque of .01Nm have to be fired. The model here developed has been applied to this mission that is characterized by the following parameters $\Omega = .52 \frac{rad}{sec.}$, $A = 700Kgm^2$, $B = C = 500Kgm^2$. The low value of the thrust combined with a fairly good engines specific impulse of 500s makes the need for optimizing this manoeuvre quite irrelevant from a mass consumption point of view, it was anyway important for this case to demonstrate that the reorientation could be achieved in a relatively short



Figure 3: Optimal trajectory (minimum time/energy)

strategy	time requested	$\int \mathbf{u} \cdot \mathbf{u}$
despin-rotate-spin	2276sec.	$0.2122N^2m^2sec$
initial guess	400sec.	$0.03399N^2m^2sec$
time optimal	304.3sec.	$0.04262N^2m^2sec$
time-power optimal	304.5sec.	$0.0285N^2m^2sec$

Table 1: Comparison between the different strategies for the POP case.

time.

Different results are reported in table 1 for four different strategies: *time optimal*, *time-power optimal*, *initial guess*, *despin-rotate-spin*. We conclude by showing (figure 4), in a case in which the maximum torque was set to .1*Nm*, how the consideration of the mixed optimal problem leads to a non bang-bang control. It is of interest to note that two of the control components remain unaltered.

Conclusions

A direct transcription method has been proposed to solve the optimal problem of large reorientations of a spinning gyrostat. The defects have been written by using a Runge-Kutta formula that still allows to consider a finer mesh for the controls and



Figure 4: Comparison between time optimal strategy and time-power optimal strategy in a POP-like case

to obtain a sparse Non Linear Programming problem. An initial guess is proposed together with the consideration of a mixed time-power cost function. The results show how the mixed time-power cost function returns good solutions both in terms of time and of power representing therefore a viable multi-objective optimization technique. Results are also presented in the case of a mission concept under study by the Advanced Concept Team in ESTEC.

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